



NASA - In Saturn's Shadow http://www.nasa.gov/mission_pages/cassini/multimedia/pia08329.html

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- Regulated by conservation of mass and angular momentum
- Mass conservation in 3D:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- Momentum conservation in 3D:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u} - \mathbf{T}) = -\rho \nabla \Phi$$

- Φ is axisymmetric gravitational potential we considered for orbits (later, allow non-axisymmetric potential and tidal torque)
- \mathbf{T} represents collective effects, including self-gravity

Evolution of an accretion disc

- Write in cylindrical polar coordinates (r, ϕ, z)
- Reduce from 3D to 1D
- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \phi} (\rho u_\phi) + \frac{\partial}{\partial z} (\rho u_z) = 0$$

- Integrate $r \int_{-\infty}^{\infty} \int_0^{2\pi} \cdot d\phi dz$ over cylinder of radius r

$$\Rightarrow \frac{\partial \mathcal{M}}{\partial t} + \frac{\partial \mathcal{F}}{\partial r} = 0 \quad \text{assuming no vertical mass loss or gain}$$

- 1D mass density $\mathcal{M}(r, t) = \int_{-\infty}^{\infty} \int_0^{2\pi} \rho r d\phi dz$

- Radial mass flux $\mathcal{F}(r, t) = \int_{-\infty}^{\infty} \int_0^{2\pi} \rho u_r r d\phi dz$

Evolution of an accretion disc

- Angular momentum conservation: (r, ϕ, z)

$$\frac{\partial}{\partial t}(\rho u_\phi) + \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 (\rho u_r u_\phi - T_{r\phi})] + \frac{1}{r} \frac{\partial}{\partial \phi} (\rho u_\phi^2 - T_{\phi\phi}) + \frac{\partial}{\partial z} (\rho u_\phi u_z - T_{\phi z}) = 0$$

- Integrate $r^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \cdot d\phi dz$ over cylinder of radius r

- Assume that $ru_\phi = h(r)$ from orbital dynamics
(examine this assumption later)

$$\Rightarrow \frac{\partial}{\partial t}(\mathcal{M}h) + \frac{\partial}{\partial r}(\mathcal{F}h + \mathcal{G}) = 0 \quad \text{assuming no vertical loss or gain}$$

- Internal torque $\mathcal{G}(r, t) = - \int_{-\infty}^{\infty} \int_0^{2\pi} r^2 T_{r\phi} d\phi dz$

- Since h depends only on r :

$$\frac{\partial \mathcal{M}}{\partial t} + \frac{\partial \mathcal{F}}{\partial r} = 0$$

$$\frac{\partial \mathcal{M}}{\partial t} h + \frac{\partial}{\partial r} (\mathcal{F}h + \mathcal{G}) = 0$$

- Eliminate \mathcal{M} :

$$\mathcal{F} \frac{dh}{dr} + \frac{\partial \mathcal{G}}{\partial r} = 0 \quad \text{which determines } \mathcal{F}$$

$$\frac{\partial \mathcal{M}}{\partial t} = \frac{\partial}{\partial r} \left[\left(\frac{dh}{dr} \right)^{-1} \frac{\partial \mathcal{G}}{\partial r} \right]$$

- Interpretation:

- angular momentum determines orbital radius
- angular momentum transport determines mass evolution

- More usual notation:

$$\mathcal{M} = 2\pi r \Sigma$$

$$\mathcal{F} = 2\pi r \Sigma \bar{u}_r$$

$$\mathcal{G} = -2\pi \bar{\nu} \Sigma r^3 \frac{d\Omega}{dr}$$

$$\Sigma(r, t) \quad \text{surface density}$$

$$\bar{u}_r(r, t) \quad \text{mean radial velocity}$$

$$\bar{\nu}(r, t) \quad \text{mean effective kinematic viscosity}$$

- Equivalent to:

$$\Sigma = \int_{-\infty}^{\infty} \langle \rho \rangle dz$$

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdot d\phi$$

$$\Sigma \bar{u}_r = \int_{-\infty}^{\infty} \langle \rho u_r \rangle dz$$

$$\bar{\nu} \Sigma = \int_{-\infty}^{\infty} \langle \mu \rangle dz$$

$$T_{r\phi} = \mu r \frac{d\Omega}{dr}$$

- Then obtain:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{dh}{dr} \right)^{-1} \frac{\partial}{\partial r} \left(-\bar{\nu} \Sigma r^3 \frac{d\Omega}{dr} \right) \right]$$

- Keplerian disc:

$$\Omega \propto r^{-3/2}$$

$$h = r^2 \Omega \propto r^{1/2}$$

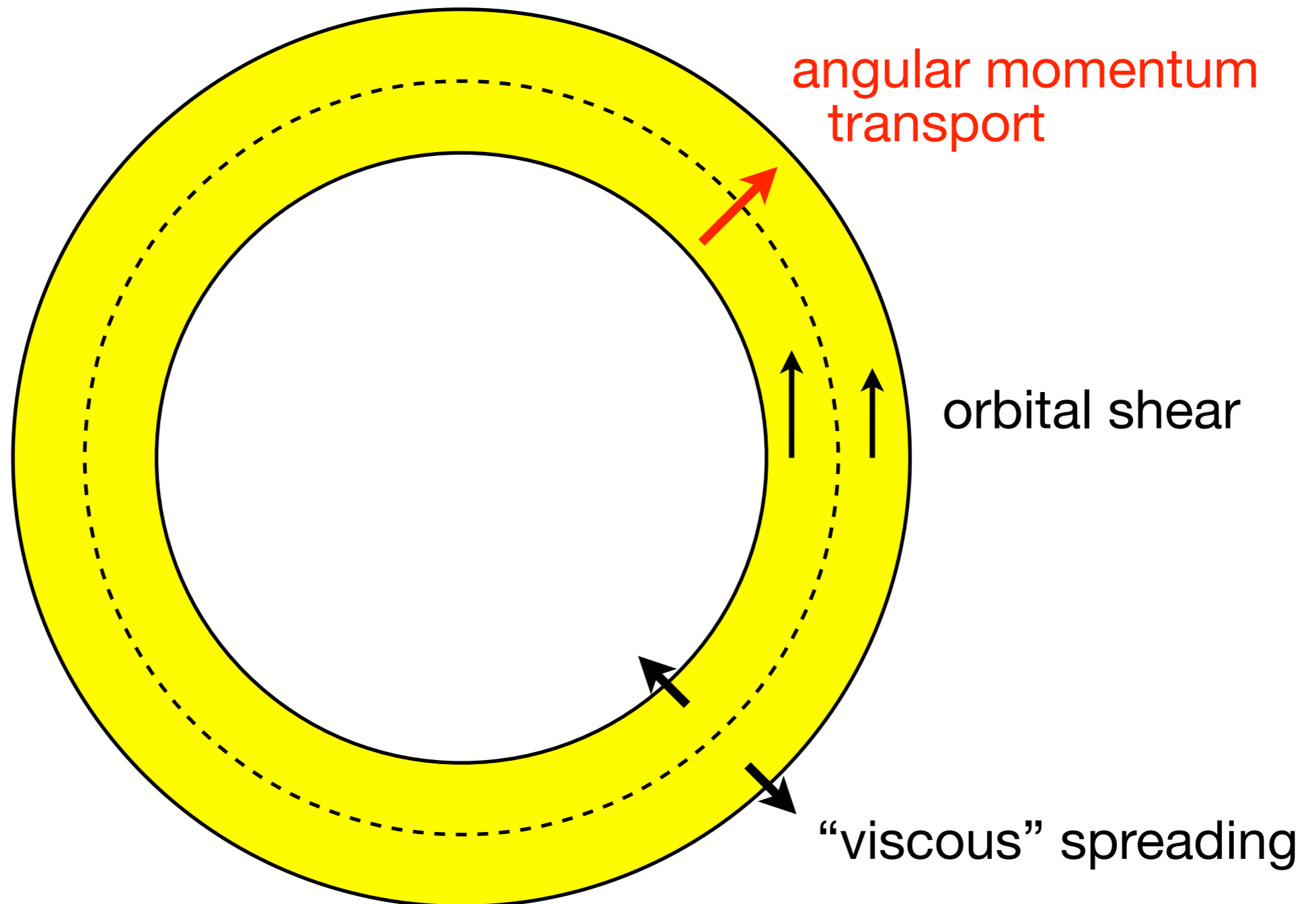
$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (r^{1/2} \bar{\nu} \Sigma) \right]$$

diffusion equation
for surface density

- Interpretation:

$$\frac{d\Omega}{dr} < 0$$

$$\frac{dh}{dr} > 0$$

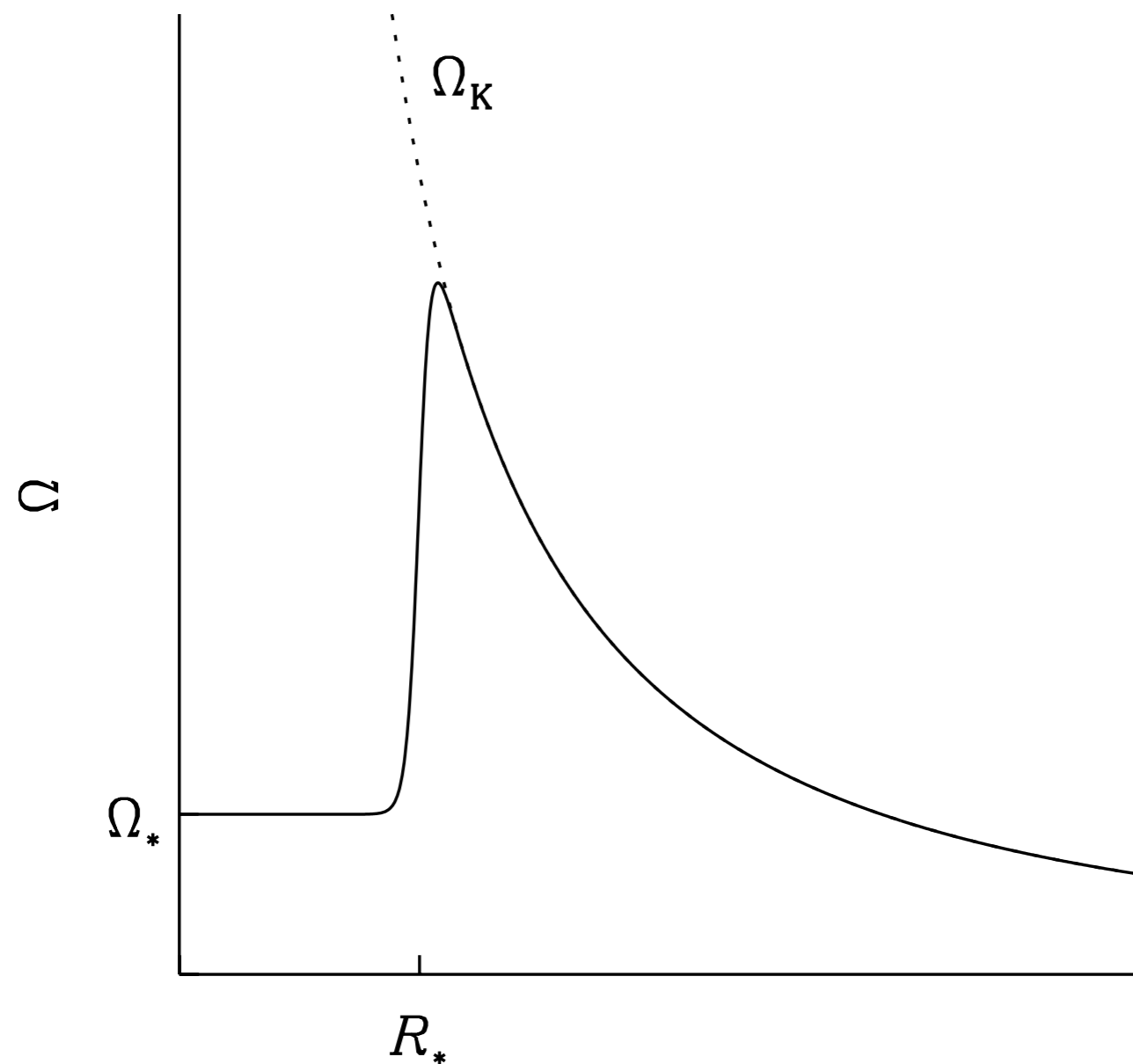


- Inner boundary condition:
 - Important because spreading ring will reach $r = 0$ in finite time
 - Depends on nature of central object
- Star with negligible magnetic field:
 - Disc may extend down to stellar surface
 - Star usually rotates at only a fraction of Keplerian rate:

$$\Omega_* < \left(\frac{GM}{R_*^3} \right)^{1/2}$$

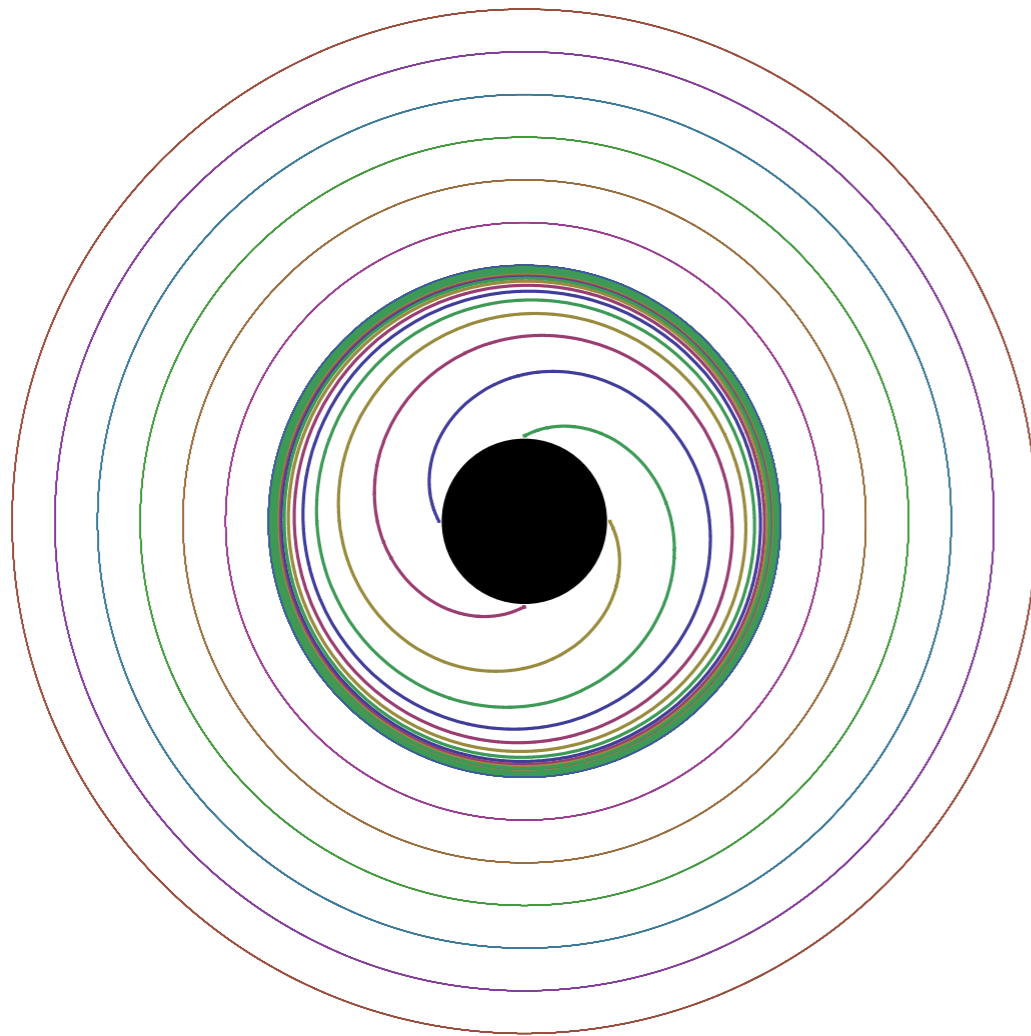
(star is supported mainly by pressure)

- Angular velocity makes a rapid adjustment from Keplerian to stellar in a viscous boundary layer



- Usual argument: $\frac{d\Omega}{dr} = 0$ at $r = r_{\text{in}} (\approx R_*)$ so $\mathcal{G} = 0$ there

- Black hole:
 - Circular orbits are unstable close to event horizon



Non-rotating black hole:

$$\Omega = \left(\frac{GM}{r^3} \right)^{1/2}$$

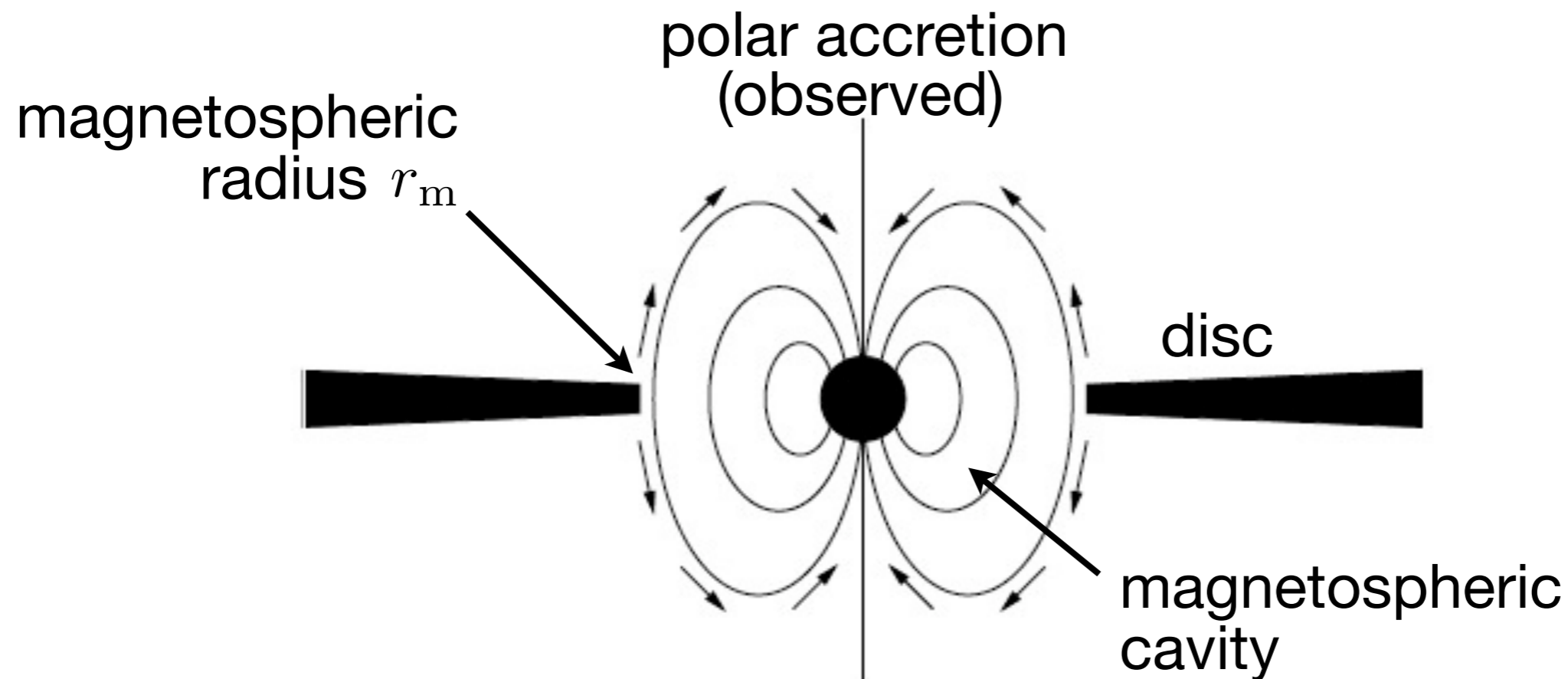
$$\kappa^2 = \Omega^2 \left(1 - \frac{6GM}{c^2 r} \right)$$

Orbits unstable for

$$r < \frac{6GM}{c^2} = 3r_S$$

- Rapid inspiral from r_{ms}
- $\bar{u}_r \uparrow$ and $\Sigma \downarrow$ rapidly, so effectively $\mathcal{G} \approx 0$ at $r_{\text{in}} \approx r_{\text{ms}}$

- Star with significant magnetic field (e.g. dipole):



- \mathcal{G} not necessarily 0 at $r_{in} \approx r_m$
- Star can exert torque on disc
- Depends on magnetic coupling (complicated problem)

- Mathematically, we may let $r_{\text{in}} \rightarrow 0$
- To allow mass flux at origin but no torque:

$$\left. \begin{array}{l} \mathcal{F} \rightarrow \text{cst} \\ \mathcal{G} \rightarrow 0 \end{array} \right\} \text{ as } r \rightarrow 0$$

- To allow torque at origin:

$$\mathcal{G} \rightarrow \text{cst} \quad \text{as } r \rightarrow 0$$

- For a Keplerian disc,

$$\begin{aligned} \mathcal{F} &\propto r \Sigma \bar{u}_r \propto r^{1/2} \frac{\partial}{\partial r} (r^{1/2} \bar{v} \Sigma) \\ \mathcal{G} &\propto r^{1/2} \bar{v} \Sigma \end{aligned}$$

- First case (no torque): $\bar{v} \Sigma \rightarrow \text{cst} \quad \text{as } r \rightarrow 0$
- Second case (torque): $r^{1/2} \bar{v} \Sigma \rightarrow \text{cst} \quad \text{as } r \rightarrow 0$

- Consideration of angular momentum transport processes and local vertical structure of disc (later) leads to

$$\bar{\nu} = \bar{\nu}(r, \Sigma) \quad (\text{e.g. double power law})$$

- If $\bar{\nu} = \bar{\nu}(r)$ only, diffusion equation is **linear**
- If $\bar{\nu} = \bar{\nu}(r, \Sigma)$, diffusion equation is **nonlinear**
- Usually solve as initial-value problem or look at steady solutions
- Linear case can be treated using Green's function $G(r, s, t)$:
- Solution of diffusion equation with initial condition

$$\Sigma(r, 0) = \delta(r - s) \quad (\text{i.e. very narrow ring})$$

- Then solution for any initial condition $\Sigma_0(r)$ is

$$\Sigma(r, t) = \int_0^{\infty} G(r, s, t) \Sigma_0(s) ds$$

- Can calculate $G(r, s, t)$ in terms of Bessel functions for any power law $\bar{\nu} \propto r^p$ and any boundary conditions
- Classic example: (Lynden-Bell & Pringle 1974)

$$\bar{\nu} = \text{cst}$$

$$\mathcal{G} = 0 \text{ inner BC at } r = 0$$

no outer boundary

$$G(r, s, t) = \frac{r^{-1/4} s^{5/4}}{6\bar{\nu}t} \exp\left[-\frac{r^2 + s^2}{12\bar{\nu}t}\right] I_{1/4}\left(\frac{rs}{6\bar{\nu}t}\right)$$

modified Bessel function



$$I_{1/4}\left(\frac{rs}{6\bar{\nu}t}\right)$$

(omit derivation)

- Becomes more asymmetrical as t increases
- As $t \rightarrow \infty$, all mass accreted by central object and all angular momentum carried to $r = \infty$ by negligible mass

- Easier example: suppose $\bar{v} = Ar$ with $A = \text{const}$:

- Let $y = \left(\frac{4r}{3A}\right)^{1/2}$ and $g = r^{1/2}\bar{v}\Sigma = Ar^{3/2}\Sigma$ to obtain

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2} \quad \text{classical diffusion equation}$$

- Spreading-ring solution with zero torque ($g = 0$) at centre:

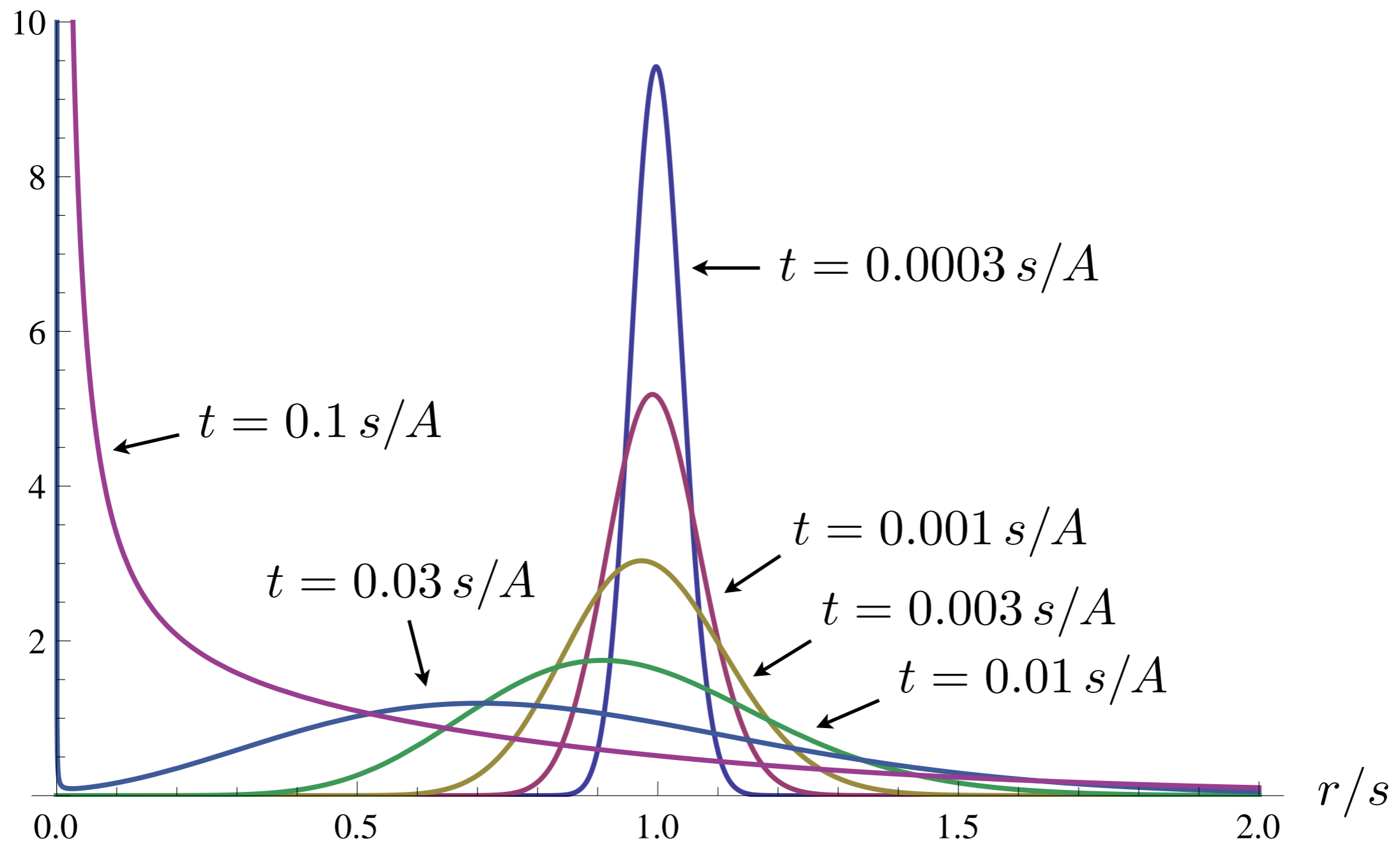
$$g \propto t^{-1/2} \left\{ \exp\left[-\frac{(y - y_0)^2}{4t}\right] - \exp\left[-\frac{(y + y_0)^2}{4t}\right] \right\}$$

- Thus

$$G(r, s, t) = \frac{r^{-3/2}t^{-1/2}}{(3\pi A)^{1/2}} \exp\left[-\frac{(r + s)}{3At}\right] \sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$

- Remaining mass $\propto \int_0^\infty G(r, s, t) r dr = \text{erf}\left[\left(\frac{s}{3At}\right)^{1/2}\right]$

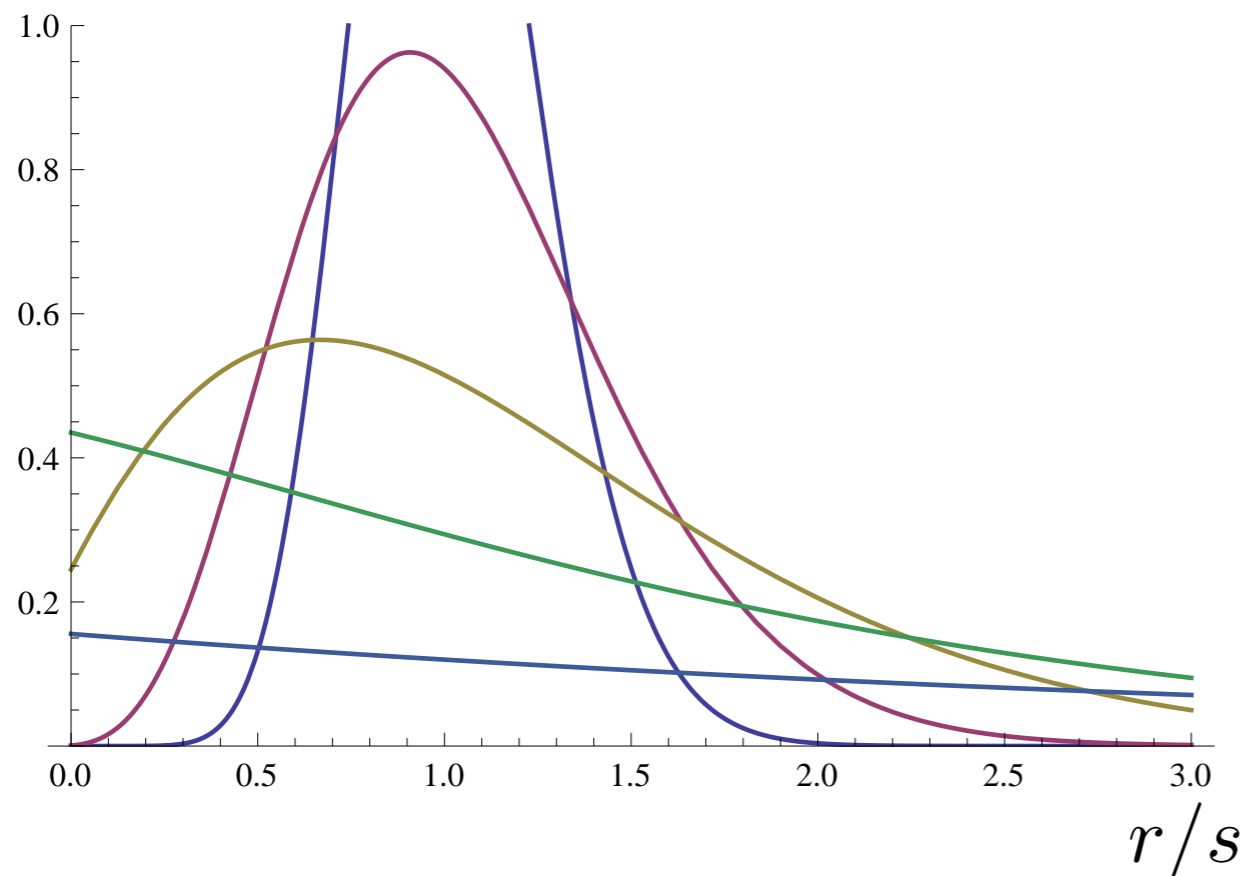
$$G(r, s, t) = \frac{r^{-3/2} t^{-1/2}}{(3\pi A)^{1/2}} \exp\left[-\frac{(r+s)}{3At}\right] \sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$



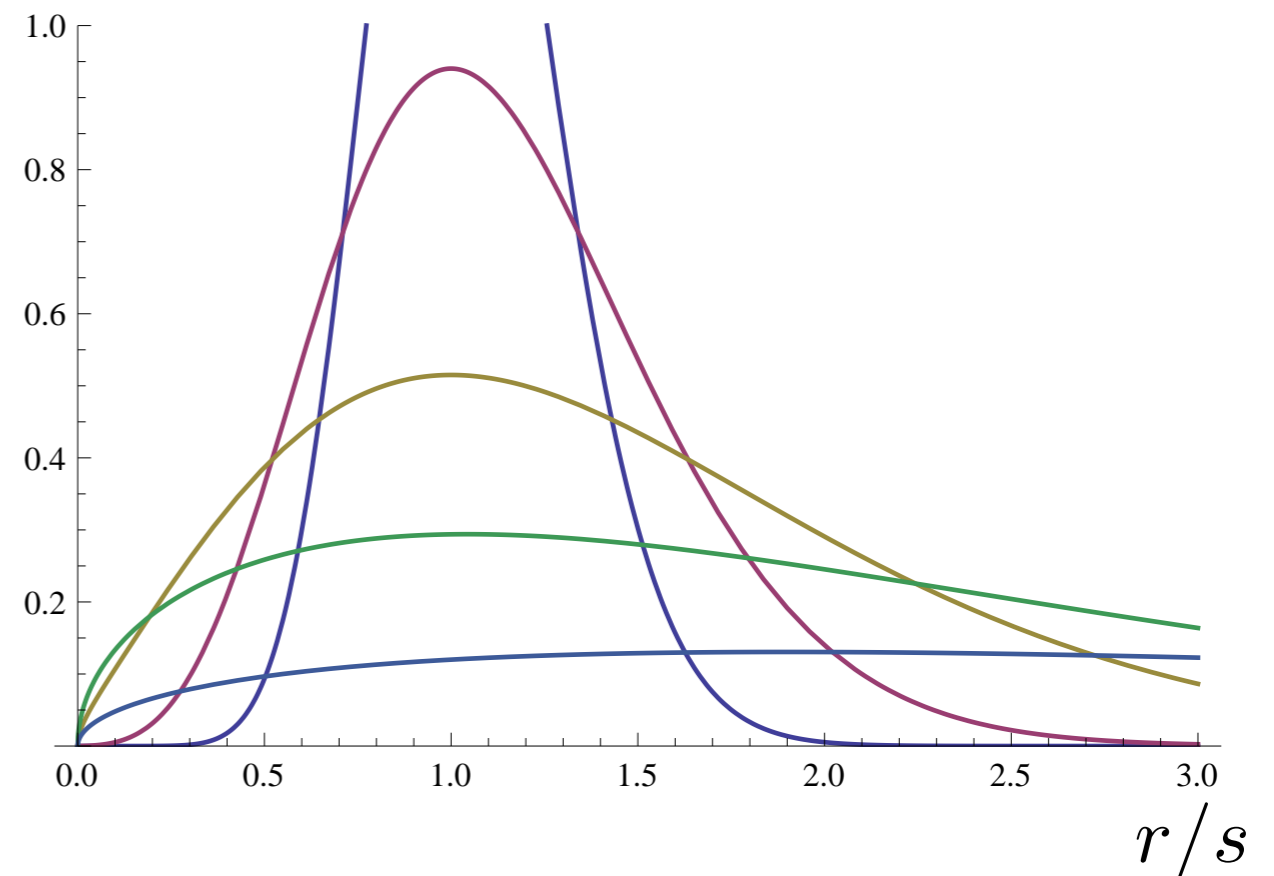
Analysis of the diffusion equation

$$G(r, s, t) = \frac{r^{-3/2} t^{-1/2}}{(3\pi A)^{1/2}} \exp\left[-\frac{(r+s)}{3At}\right] \sinh\left[\frac{2(rs)^{1/2}}{3At}\right]$$

$Gr \propto$ mass / unit radius



$Gr^{3/2} \propto$ ang mom / unit radius



$$t = \{0.01, 0.03, 0.1, 0.3, 1\} s/A$$

Analysis of the diffusion equation

- Nonlinear case with $\bar{\nu}\Sigma \propto r^p \Sigma^q$ and $r_{\text{in}} \rightarrow 0$:
- No intrinsic length-scale
- Special algebraic similarity solutions
- Generally attract solutions of initial-value problem
- But if $q < 0$, viscous instability occurs (see later)

- Example: $\bar{v} = Ar^2\Sigma$
- Two similarity solutions of the form

$$\Sigma = \sigma(t) \left\{ \left[\frac{R(t)}{r} \right]^a - 1 \right\}, \quad r \leq R(t)$$

- Solution 1:

$$a = 1 \quad \sigma = \frac{1}{15At}$$

$$R = Ct^{2/5}$$

$$\bar{u}_r = -\frac{(R - 5r)}{10t}$$

$$r = R - ct^{1/2} \quad (\text{trajectories})$$

- Solution 2:

$$a = 5/4 \quad \sigma = \frac{1}{15At}$$

$$R = Ct^{1/2}$$

$$\bar{u}_r = \frac{r}{2t}$$

$$r = ct^{1/2}$$

- Accretion disc: almost all mass accreted in finite time (no torque at centre)

- Decretion disc: almost all mass expelled requires torque at centre)

- Steady discs

$$\mathcal{F} = \text{cst} = -\dot{M}$$

\dot{M} = mass accretion rate

(but always neglect slow increase of M)

$$\mathcal{F} \frac{dh}{dr} + \frac{d\mathcal{G}}{dr} = 0 \quad \Rightarrow \quad -\dot{M}h + \mathcal{G} = \text{cst}$$

- If $\mathcal{G} = 0$ at $r = r_{\text{in}}$, solution is $\mathcal{G} = \dot{M}(h - h_{\text{in}})$

- In Keplerian case (recall $\mathcal{G} = -2\pi\bar{\nu}\Sigma r^3 \frac{d\Omega}{dr}$)

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{r_{\text{in}}}{r} \right)^{1/2} \right]$$

- If we know $\bar{\nu}(r, \Sigma)$, we can solve for $\Sigma(r)$

- Other solutions:

- Non-accreting disc: $\mathcal{F} = 0, \mathcal{G} = \text{cst}$
 - Decretion disc: $\mathcal{F} > 0$
- } require torque from central object

- Note that $\mathcal{F} < 0$ requires $\frac{d\mathcal{G}}{dr} > 0$

- Energy balance in steady disc:

- Energy dissipation rate per unit volume $= T_{r\phi} r \frac{d\Omega}{dr} = \mu \left(r \frac{d\Omega}{dr} \right)^2$

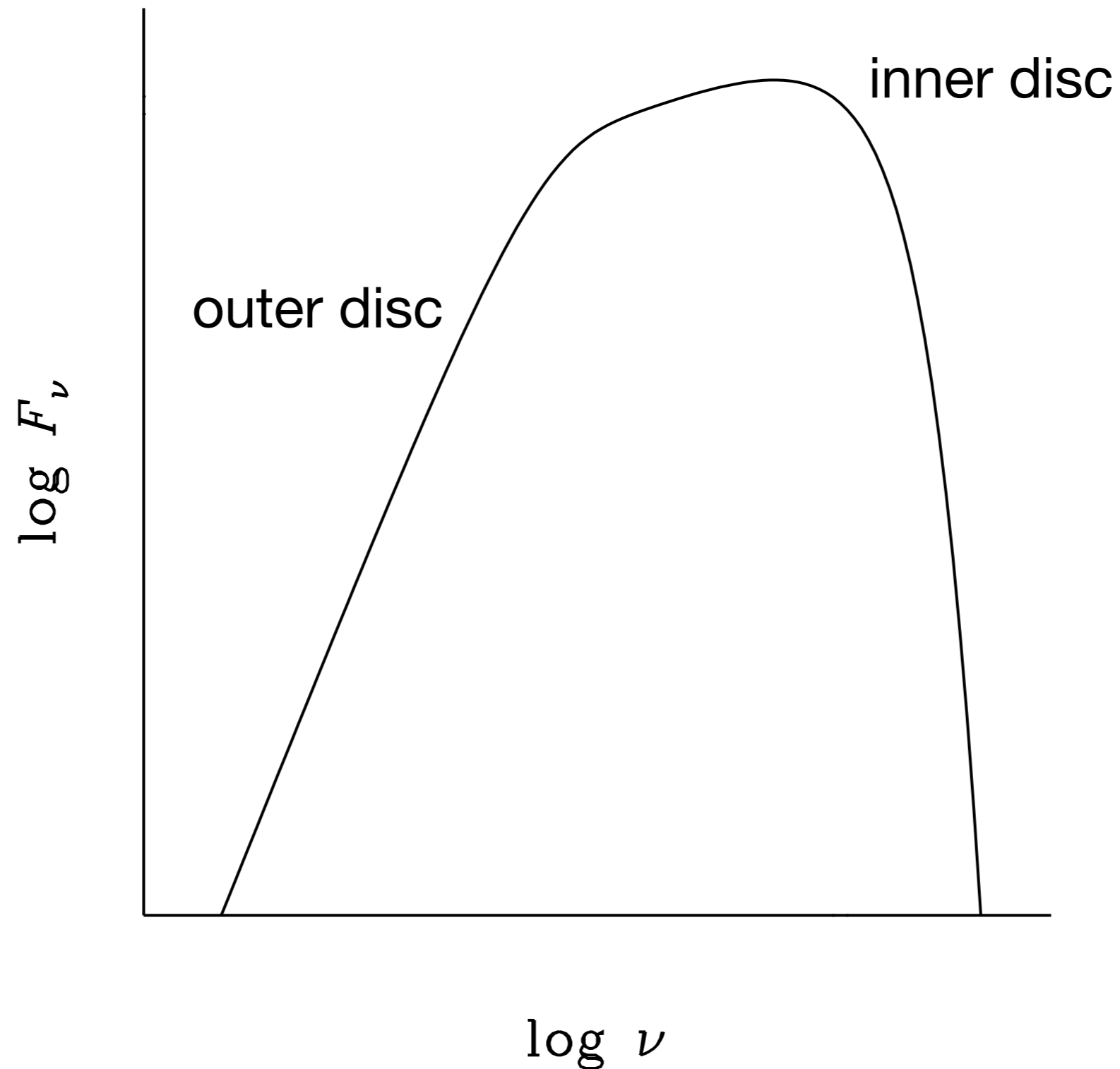
- Energy emission rate per unit area (from each face of disc)

$$= \frac{1}{2} \bar{\nu} \Sigma \left(r \frac{d\Omega}{dr} \right)^2$$

- Effective blackbody temperature $T_{\text{eff}}(r)$ given by

$$\sigma T_{\text{eff}}^4 = \frac{9}{8} \bar{\nu} \Sigma \Omega^2 = \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{r_{\text{in}}}{r} \right)^{1/2} \right]$$

- Typical (theoretical) spectrum of emitted radiation:



- Total luminosity of disc ($r_{\text{in}} < r < \infty$)

$$L_{\text{disc}} = \int_{r_{\text{in}}}^{\infty} \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{r_{\text{in}}}{r} \right)^{1/2} \right] 2\pi r \, dr = \frac{1}{2} \frac{GM\dot{M}}{r_{\text{in}}}$$

= rate of release of orbital binding energy ($\infty \rightarrow r_{\text{in}}$)

= $\frac{1}{2}$ rate of release of potential energy

(remaining kinetic energy released in boundary layer
if $\Omega_* = 0$)

- Vertical hydrostatic equilibrium
 - Dominant balance (for non-self-gravitating gas disc)

$$0 = -\rho \frac{\partial \Phi}{\partial z} - \frac{\partial p}{\partial z}$$

- Expand potential

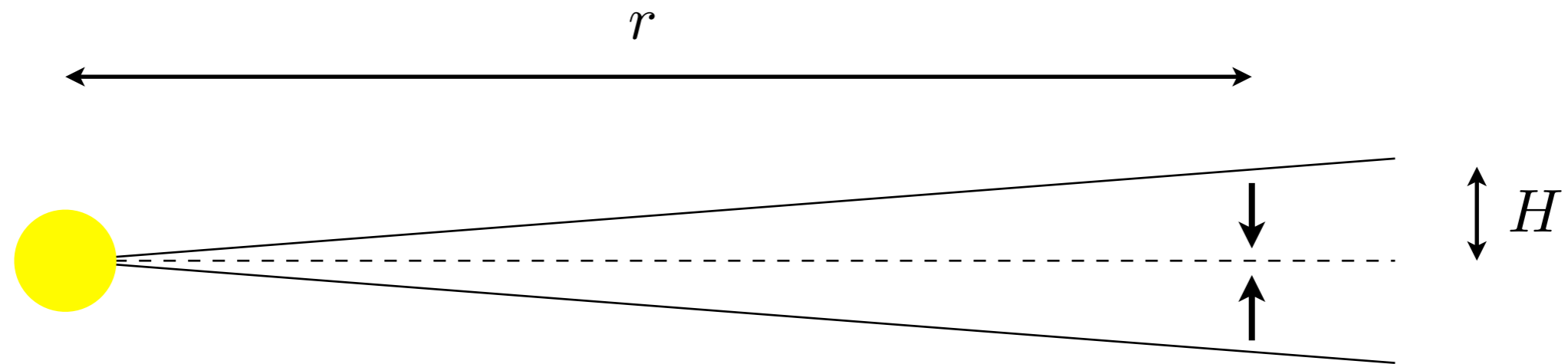
$$\Phi(r, z) = \Phi(r, 0) + \frac{1}{2} \Phi_{zz}(r, 0) z^2 + \dots$$

- So for a thin disc

$$\frac{\partial \Phi}{\partial z} \approx \Omega_z^2 z \quad (\Omega_z = \Omega \text{ for Keplerian disc})$$

- Equation of vertical hydrostatic equilibrium

$$\frac{\partial p}{\partial z} \approx -\rho \Omega_z^2 z$$



- Disc supported against gravity by pressure in vertical direction
- Centrifugal support in radial direction

- Order-of-magnitude estimates and time-scales
 - Simple scaling arguments (\sim), omitting factors of order unity
 - Important dimensionless parameter:

$$\frac{H}{r} \quad (\text{angular semi-thickness, aspect ratio})$$

- For a thin disc: $\frac{H}{r} \ll 1$

- H may represent: $\left\{ \begin{array}{l} \text{true semi-thickness} \\ \text{height of photosphere above midplane} \\ \text{measure of density scale-height} \end{array} \right.$

- From vertical hydrostatic equilibrium:

$$\frac{p}{H} \sim \rho \Omega^2 H \quad \Rightarrow \quad c_s \sim \Omega H$$

$$c_s = \left(\frac{p}{\rho} \right)^{1/2}$$

isothermal
sound speed

- (Effective) viscosity:

- Dimensional argument: $[\mu] = ML^{-1}T^{-1} = \left[\frac{p}{\Omega} \right]$

- Write $\mu = \frac{\alpha p}{\Omega}$ in terms of dimensionless viscosity α
 (“alpha viscosity prescription”, Shakura & Sunyaev 1973)

- (Mean) kinematic viscosity $\bar{\nu} \sim \frac{\mu}{\rho} \sim \frac{\alpha c_s^2}{\Omega} \sim \alpha c_s H$

- Analogous to kinetic theory of molecular transport:

$$\nu \sim v \ell \quad \text{for mean speed } v \text{ and mean free path } \ell$$

- Molecular viscosity usually negligible for astrophysical discs

- Similar estimate applies to effective “eddy viscosity” of turbulence
of mean turbulent speed v and correlation length ℓ

- Assuming that $v \lesssim c_s$ (subsonic turbulence) and $\ell \lesssim H$,
expect that $\alpha \lesssim 1$

- Stress in a Keplerian disc: $T_{r\phi} = \mu r \frac{d\Omega}{dr} = -\frac{3}{2}\alpha p$
- Alternative version of alpha prescription: $T_{r\phi} = -\alpha p$
- Whatever process gives rise to stress, scales with local pressure
- Reasonable assumption for local processes (see later)

- Three important characteristic time-scales:

- Dynamical time-scale:

$$t_{\text{dyn}} \sim \frac{1}{\Omega} \quad (\text{orbital motion}) \quad \sim \frac{H}{c_s} \quad (\text{hydrostatic})$$

- Viscous time-scale:

$$t_{\text{visc}} \sim \frac{r^2}{\bar{\nu}} \sim \alpha^{-1} \left(\frac{H}{r} \right)^{-2} t_{\text{dyn}} \quad (\text{radial motion})$$

- Thermal time-scale:

$$t_{\text{th}} \sim \frac{pH}{\bar{\nu}\Sigma\Omega^2} \sim \frac{c_s^2}{\bar{\nu}\Omega^2} \sim \frac{H^2}{\bar{\nu}} \sim \alpha^{-1} t_{\text{dyn}} \quad (\text{thermal balance})$$

- For a thin disc with $\alpha < 1$: $t_{\text{dyn}} < t_{\text{th}} \ll t_{\text{visc}}$
- All three time-scales usually increase rapidly with r

- Mach number of orbital motion:

$$\text{Ma} \sim \frac{r\Omega}{c_s} \sim \left(\frac{H}{r}\right)^{-1}$$

- Characteristic radial velocity due to viscosity:

$$|\bar{u}_r| \sim \frac{\bar{v}}{r} \sim \alpha \left(\frac{H}{r}\right) c_s$$

- For a thin disc,

$$|\bar{u}_r| \ll c_s \ll r\Omega$$

orbital motion: highly supersonic

accretion flow: highly subsonic

- Corrections to orbital motion:
 - Contribution of radial pressure gradient:

$$\frac{\partial p}{\partial r} / \rho r \Omega^2 \sim \frac{c_s^2}{r^2 \Omega^2} \sim \left(\frac{H}{r} \right)^2$$

- Vertical variations of radial gravitational acceleration $\sim \left(\frac{H}{r} \right)^2$

- Other terms, e.g. $\rho u_r \frac{\partial u_r}{\partial r}$, are smaller

- Conclude that $u_\phi = r\Omega \left[1 + O \left(\frac{H}{r} \right)^2 \right]$

i.e. fluid velocity close to orbital velocity of test particle

- In general, thin-disc approximations involve fractional errors $O \left(\frac{H}{r} \right)^2$ and a formal asymptotic treatment is possible

Typical values of H/r :

- protoplanetary discs : 0.05 – 0.1
- binary stars : 0.01 – 0.02
- active galactic nuclei : 0.001
- planetary rings : 0.0000001

- Barotropic models:
- Vertical hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\rho \Omega_z^2 z$$

- Can be solved if pressure is a known function of density (physical arguments, or just for analytical convenience)
- Vertically isothermal model:

$$p = c_s^2 \rho \quad c_s \text{ independent of } z$$

- Solution:

$$\rho(r, z) = \rho_0(r, 0) e^{-z^2/2H^2}$$

$$H = \frac{c_s}{\Omega_z} \quad \text{isothermal scale height}$$

- Polytropic model:

$$p = K \rho^{1+1/n} \quad K, n \text{ independent of } z$$

- Introduce (pseudo-)enthalpy:

$$w = \int \frac{dp}{\rho} = (n+1)K \rho^{1/n}$$

$$\frac{\partial w}{\partial z} = -\Omega_z^2 z$$

- Solution: $w = \frac{1}{2} \Omega_z^2 (H^2 - z^2)$ $H = \text{true semi-thickness}$

$$\rho(r, z) = \rho(r, 0) \left(1 - \frac{z^2}{H^2} \right)^n \quad (\text{vacuum above } z = H)$$

$$p(r, z) = p(r, 0) \left(1 - \frac{z^2}{H^2} \right)^{n+1}$$

Vertical disc structure

- Radiative models:
- Energy dissipated by viscosity carried away by radiation
- Radiative diffusion (optically thick disc):

$$\mathbf{F} = -\frac{16\sigma T^3}{3\kappa\rho} \nabla T \quad \kappa = \text{Rosseland mean opacity}$$

- Dominant balance in thermal energy equation (thin disc):

$$0 = \mu \left(r \frac{d\Omega}{dr} \right)^2 - \frac{\partial F_z}{\partial z}$$

- Contributions from F_r and from radial advection are smaller by $O(H/r)^2$

- Vertical structure of radiative Keplerian disc:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\mu\Omega^2$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho} \frac{\partial T}{\partial z}$$

together with:

- equation of state, e.g. $p = \frac{k\rho T}{\mu_{\text{m}}m_{\text{p}}} + \frac{4\sigma T^4}{3c}$ (ideal gas + radiation)
- viscosity prescription
- opacity function $\kappa(\rho, T)$
- boundary conditions, e.g. $\rho = p = T = 0$ at $z = H$
(or match to atmospheric model)
- Analogous to radial structure of a star

- Opacity often approximated by a power law, e.g. for ionized discs:

$$\kappa = \text{const} \approx 0.33 \text{ cm}^2 \text{ g}^{-1}$$

Thomson opacity
electron scattering
hotter regions

$$\kappa = C_{\kappa} \rho T^{-7/2}$$

$$C_{\kappa} \approx 4.5 \times 10^{24} \text{ cm}^5 \text{ g}^{-2} \text{ K}^{7/2}$$

Kramers opacity
free-free / bound-free
cooler regions

- Cooler discs: dust, molecules, ...

- Keplerian disc, alpha viscosity, gas pressure, Thomson opacity:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\alpha p\Omega$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho} \frac{\partial T}{\partial z}$$

$$p = \frac{k\rho T}{\mu_m m_p}$$

$$\Sigma = \int_{-H}^H \rho dz$$

- Order-of-magnitude treatment:

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{p}{H} \sim \rho\Omega^2 H$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\alpha p\Omega$$

$$\frac{F}{H} \sim \alpha p\Omega$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho} \frac{\partial T}{\partial z}$$

$$F \sim \frac{\sigma T^3}{\kappa\rho} \frac{T}{H}$$

$$p = \frac{k\rho T}{\mu_m m_p}$$

$$p \sim \frac{k\rho T}{\mu_m m_p}$$

$$\Sigma = \int_{-H}^H \rho dz$$

$$\Sigma \sim \rho H$$

- Algebraic solution:

$$H \sim \alpha^{1/6} \Sigma^{1/3} \Omega^{-5/6} \left(\frac{\mu_m m_p}{k} \right)^{-2/3} \left(\frac{\sigma}{\kappa} \right)^{-1/6}$$

- Viscosity:

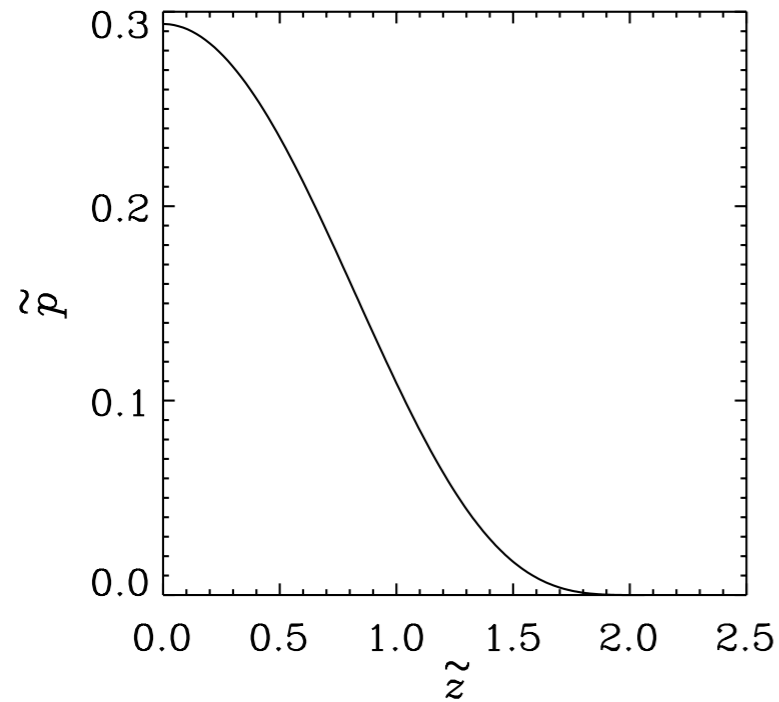
$$\bar{\nu} \sim \alpha c_s H \sim \alpha \Omega H^2 \sim \alpha^{4/3} \Sigma^{2/3} \Omega^{-2/3} \left(\frac{\mu_m m_p}{k} \right)^{-4/3} \left(\frac{\sigma}{\kappa} \right)^{-1/3}$$

- Thus $\bar{\nu} \propto r \Sigma^{2/3}$

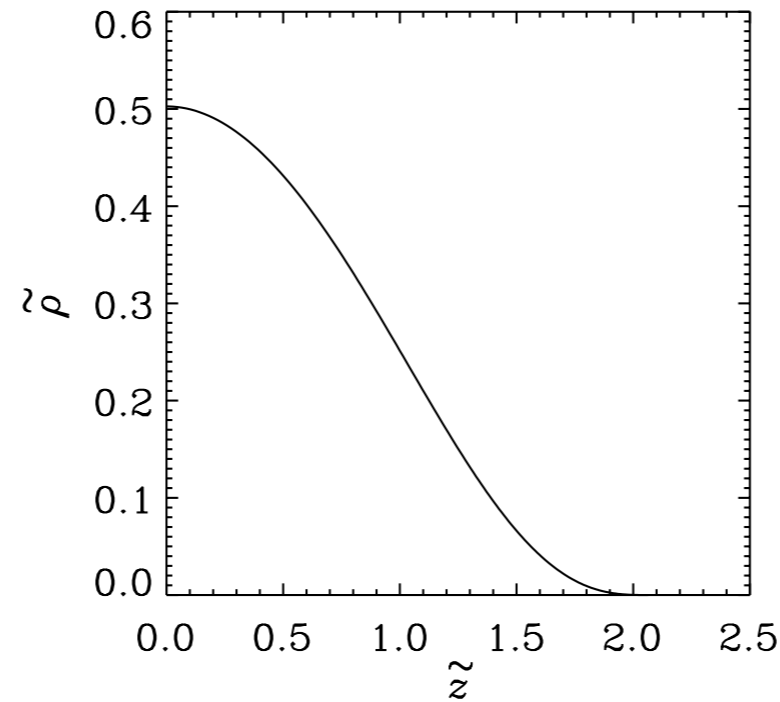
- Full treatment:

- Use order-of-magnitude treatment as a dimensional analysis
- It defines characteristic units U_H, U_ρ, U_p, U_T , etc.
- Then write $z = U_H \tilde{z}, \rho = U_\rho \tilde{\rho}(\tilde{z})$, etc. to obtain a system of dimensionless ODEs with no free parameters
- Solve numerically
- Find $\bar{\nu} = A r \Sigma^{2/3}$ with a precise coefficient A
- Power-law constitutive relations give power-law viscosity

pressure



density



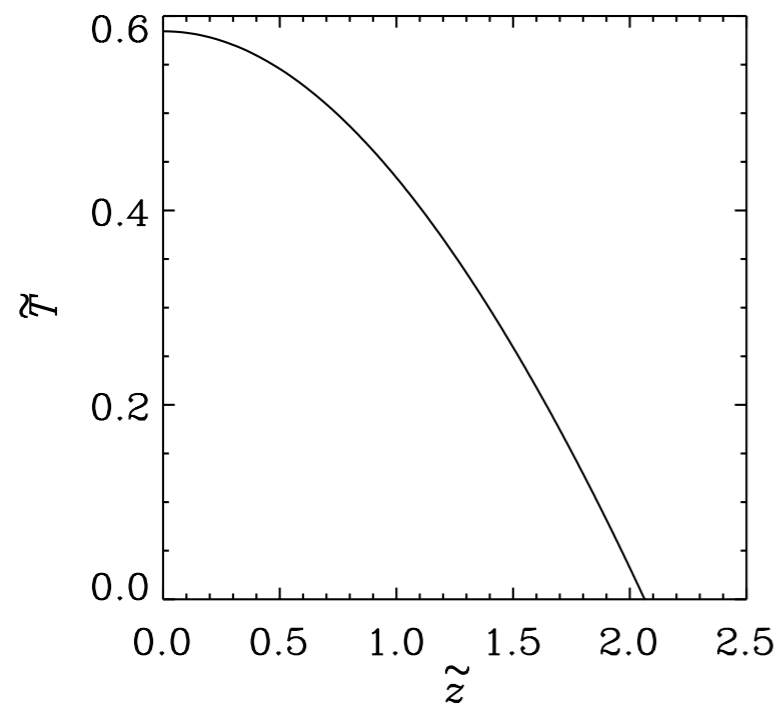
similar to
polytropic models

$$\rho \propto (H^2 - z^2)^n$$

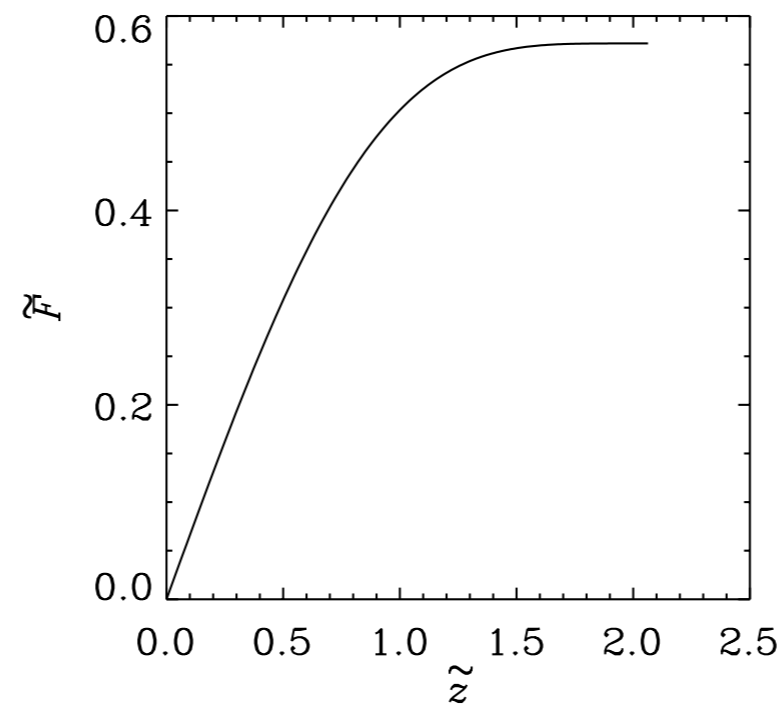
$$p \propto (H^2 - z^2)^{n+1}$$

$$T \propto (H^2 - z^2)$$

temperature



radiative flux



- A curious result:
- Thomson opacity, radiation pressure-dominated (near black holes)

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\mu\Omega^2$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho} \frac{\partial T}{\partial z}$$

$$p = \frac{4\sigma T^4}{3c}$$

- A solution is possible for only one value of the effective viscosity:

$$\mu = \frac{4c}{9\kappa} \approx 4.0 \times 10^{10} \text{ g cm}^{-1} \text{ s}^{-1}$$

(cf. water: $\mu \approx 9.0 \times 10^{-3} \text{ g cm}^{-1} \text{ s}^{-1}$)

- Steady Keplerian disc:

$$\bar{v}\Sigma = \frac{f\dot{M}}{3\pi}, \quad f = 1 - \left(\frac{r_{\text{in}}}{r}\right)^{1/2}$$

- Combine with treatment of vertical structure, e.g. Thomson opacity and gas pressure:

$$H \sim \alpha^{1/6} \Sigma^{1/3} \Omega^{-5/6} \left(\frac{\mu_{\text{m}} m_{\text{p}}}{k}\right)^{-2/3} \left(\frac{\sigma}{\kappa}\right)^{-1/6}$$

$$\bar{v} \sim \alpha c_{\text{s}} H \sim \alpha \Omega H^2 \sim \alpha^{4/3} \Sigma^{2/3} \Omega^{-2/3} \left(\frac{\mu_{\text{m}} m_{\text{p}}}{k}\right)^{-4/3} \left(\frac{\sigma}{\kappa}\right)^{-1/3}$$

- Deduce that (insensitive to poorly known value of α)

$$H \sim \alpha^{-1/10} \Omega^{-7/10} (f\dot{M})^{1/5} \left(\frac{\mu_{\text{m}} m_{\text{p}}}{k}\right)^{-2/5} \left(\frac{\sigma}{\kappa}\right)^{-1/10}$$

- Very slightly flared:

$$\frac{H}{r} \propto f^{1/5} r^{1/20}$$

- Viscous stability:
- Nonlinear diffusion equation for Keplerian disc:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (r^{1/2} \bar{\nu} \Sigma) \right] \quad \bar{\nu} = \bar{\nu}(r, \Sigma)$$

- Linearize about any given solution $\Sigma_0(r, t)$:

$$\Sigma(r, t) = \Sigma_0(r, t) + \Sigma'(r, t) \quad |\Sigma'| \ll \Sigma_0$$

$$(\bar{\nu} \Sigma)' = \frac{\partial(\bar{\nu} \Sigma)}{\partial \Sigma} \Sigma' = q \bar{\nu} \Sigma' \quad q = \frac{\partial \ln(\bar{\nu} \Sigma)}{\partial \ln \Sigma}$$

- Linearized diffusion equation:

$$\frac{\partial \Sigma'}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (r^{1/2} q \bar{\nu} \Sigma') \right]$$

- Unstable for $q < 0$: rapid growth on short length-scales

- Thermal stability:
- So far, assumed a balance between heating and cooling:

$$\frac{9}{4}\bar{\nu}\Sigma\Omega^2 = \mathcal{H} = \mathcal{C} = 2F^+$$

- Relax this assumption, but assume that $\alpha \ll 1$ so that

$$t_{\text{dyn}} \ll t_{\text{th}} \ll t_{\text{visc}}$$

- Consider behaviour on the timescale t_{th} :
 - disc is hydrostatic
 - surface density does not evolve
- By solving equations of vertical structure except thermal balance, can calculate \mathcal{H} and \mathcal{C} as functions of $(\Sigma, \bar{\nu}\Sigma)$
- In fact \mathcal{H} depends only on $\bar{\nu}\Sigma$
- Equation of thermal balance defines a curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane

- Infinitesimal perturbations:

$$d\mathcal{H} = \frac{d\mathcal{H}}{d(\bar{\nu}\Sigma)} d(\bar{\nu}\Sigma) \quad d\mathcal{C} = \frac{\partial\mathcal{C}}{\partial\Sigma} d\Sigma + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)} d(\bar{\nu}\Sigma)$$

- Along the equilibrium curve, $d\mathcal{H} = d\mathcal{C}$ and $d(\bar{\nu}\Sigma) = q\bar{\nu} d\Sigma$

$$\Rightarrow \frac{d\mathcal{H}}{d(\bar{\nu}\Sigma)} = \frac{1}{q\bar{\nu}} \frac{\partial\mathcal{C}}{\partial\Sigma} + \frac{\partial\mathcal{C}}{\partial(\bar{\nu}\Sigma)} \quad q = \frac{\partial \ln(\bar{\nu}\Sigma)}{\partial \ln\Sigma}$$

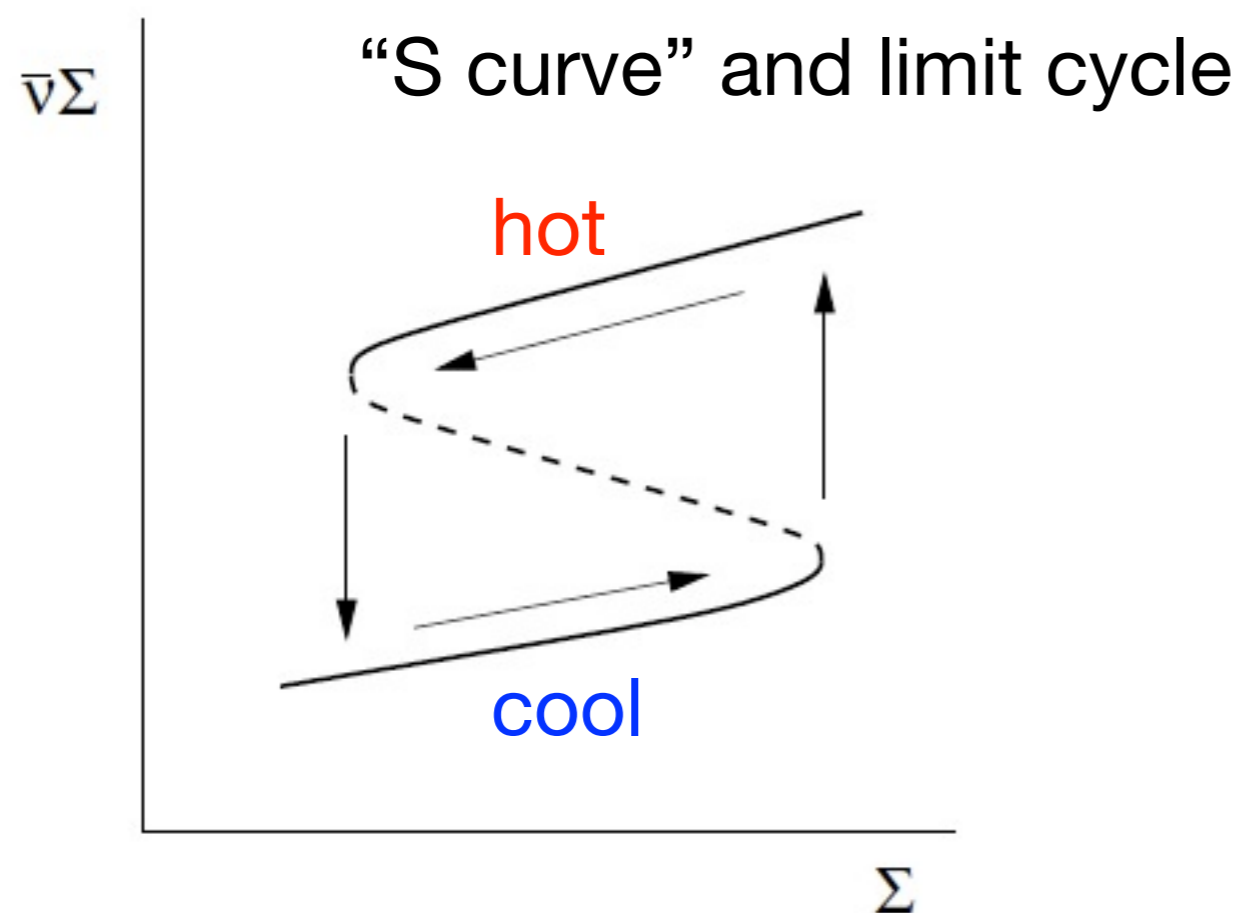
- Thermal energy content of disc per unit area $\sim pH \sim (\Omega/\alpha)\bar{\nu}\Sigma$
- If some heat is added, $\bar{\nu}\Sigma$ increases but Σ is fixed on t_{th}
- Unstable if excess heating exceeds excess cooling, i.e. if

$$\frac{d\mathcal{H}}{d(\bar{\nu}\Sigma)} > \frac{d\mathcal{C}}{d(\bar{\nu}\Sigma)} \quad \text{i.e.} \quad \frac{1}{q\bar{\nu}} \frac{\partial\mathcal{C}}{\partial\Sigma} > 0$$

- In practice $\partial\mathcal{C}/\partial\Sigma < 0$ (because, at fixed $\bar{\nu}\Sigma$, $\Sigma \propto 1/\bar{\nu} \propto 1/(\alpha T)$)
so thermal instability occurs (like viscous instability) when $q < 0$

Thermal and viscous stability

- Outbursts:
- Radiative disc with gas pressure and Thomson opacity has $\bar{\nu}\Sigma \propto r\Sigma^{5/3}$ and is viscously and thermally stable
- For cooler discs undergoing H ionization, instability can occur

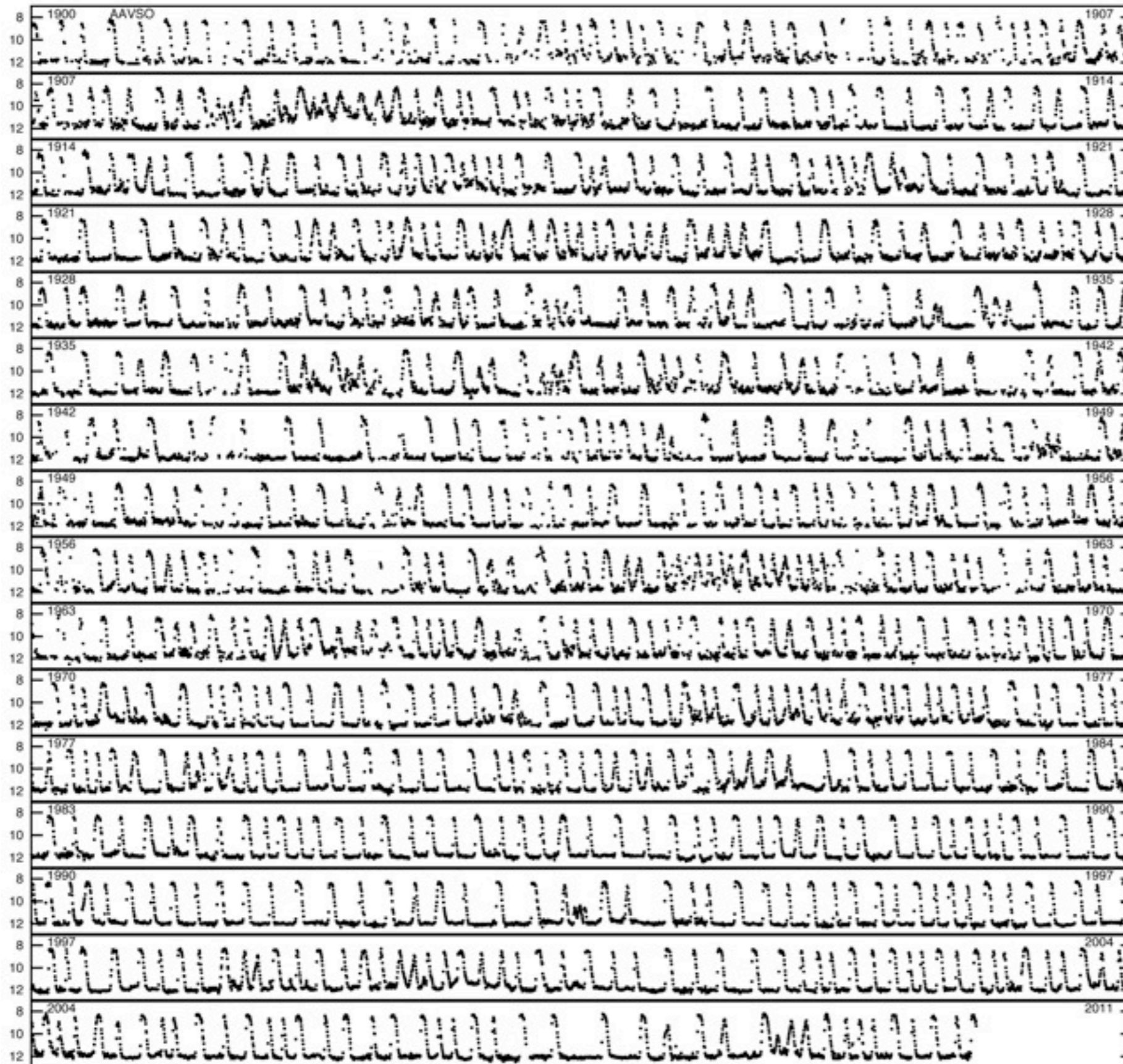


for steady accretion:

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{r_{\text{in}}}{r} \right)^{1/2} \right] \approx \frac{\dot{M}}{3\pi}$$

- Explains outbursts in many cataclysmic variables, X-ray binaries and other systems

Observations: cataclysmic variable stars



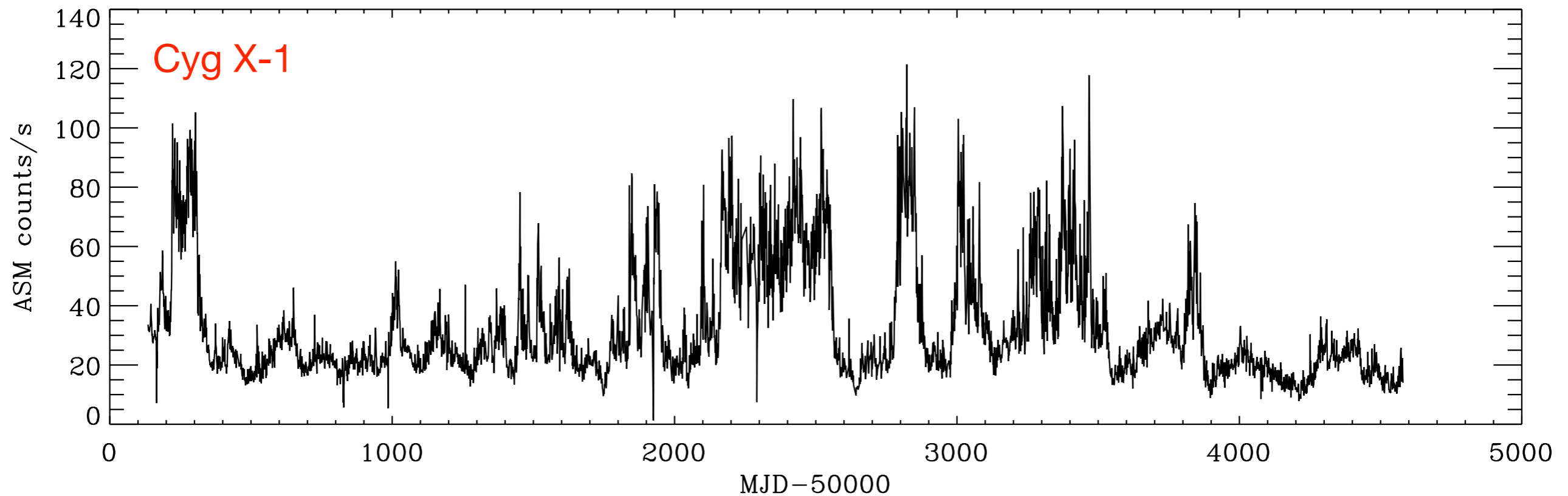
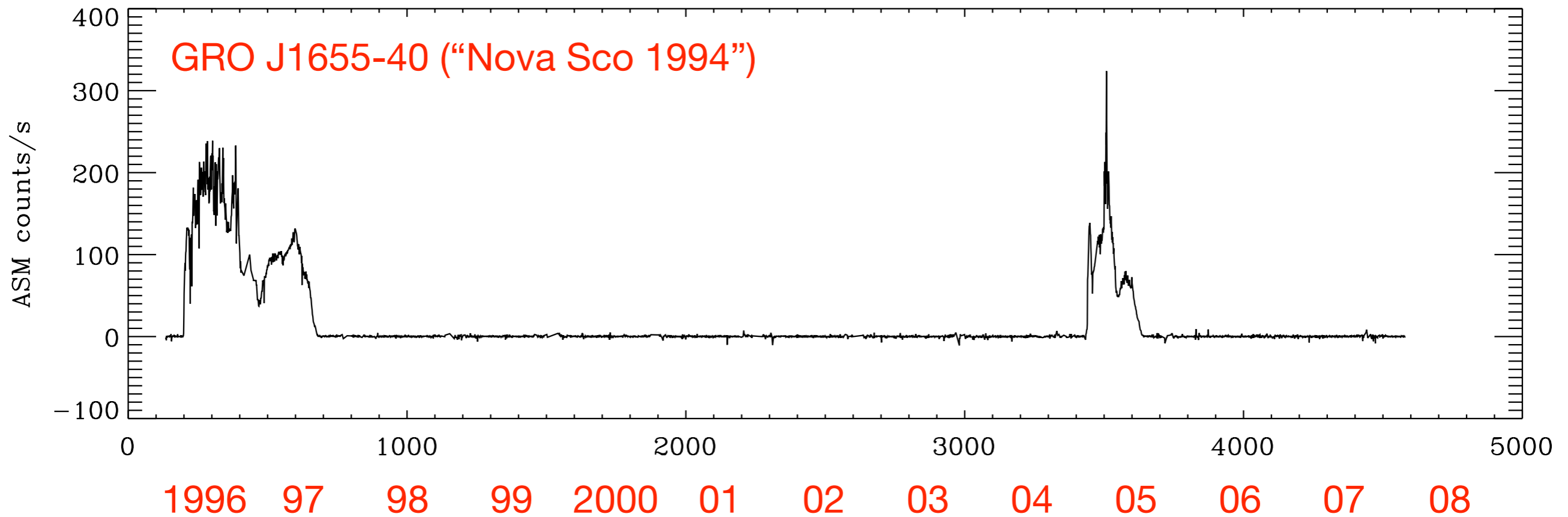
SS Cygni
dwarf nova

V magnitude
(range 12-8)
1900-2010

aavso.org

Also UV and
soft X-rays

Observations: X-ray binary stars (1960s-)



reference

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