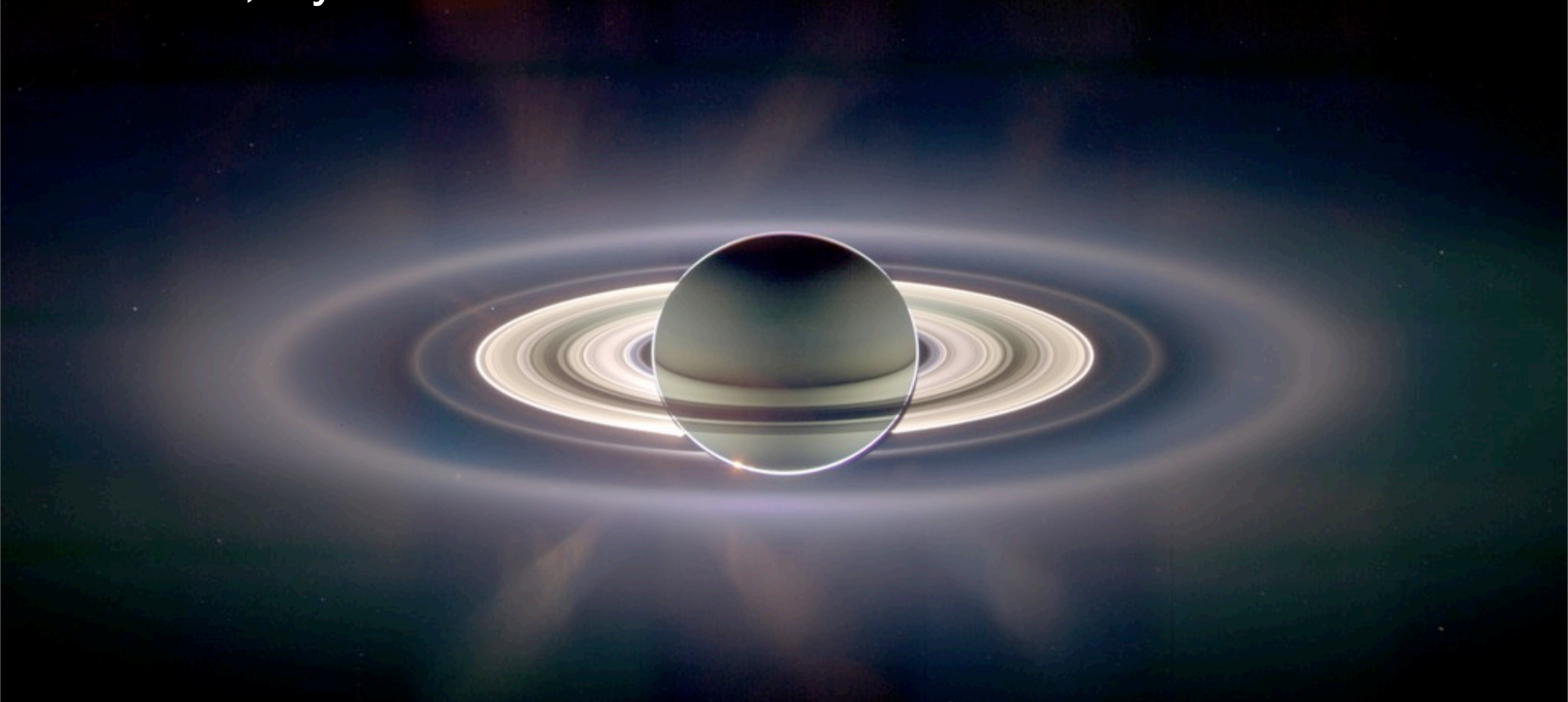
Dynamics of astrophysical discs

Lecture 5

FDEPS, Kyoto

01.12.11



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NASA - In Saturn's Shadow
http://www.nasa.gov/mission_pages/cassini/multimedia/pia08329.html

Magnetorotational instability

- Homogeneous incompressible fluid
- Local approximation (shearing sheet / box)
- 3D system, unbounded or periodic in x, y, z
- Uniform kinematic viscosity ν and magnetic diffusivity η

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}\Pi + \frac{1}{\mu_0 \rho}\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B} + \nu\nabla^2\boldsymbol{u}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \eta\nabla^2\boldsymbol{B} \qquad \Pi = p + \frac{|\boldsymbol{B}|^2}{2\mu_0}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

neglect (balanced by pressure gradient)

- Effective potential $\Phi = -\Omega S x^2 + \cancel{\tfrac{1}{2}\Omega_z^2 z^2}$
- Basic state:

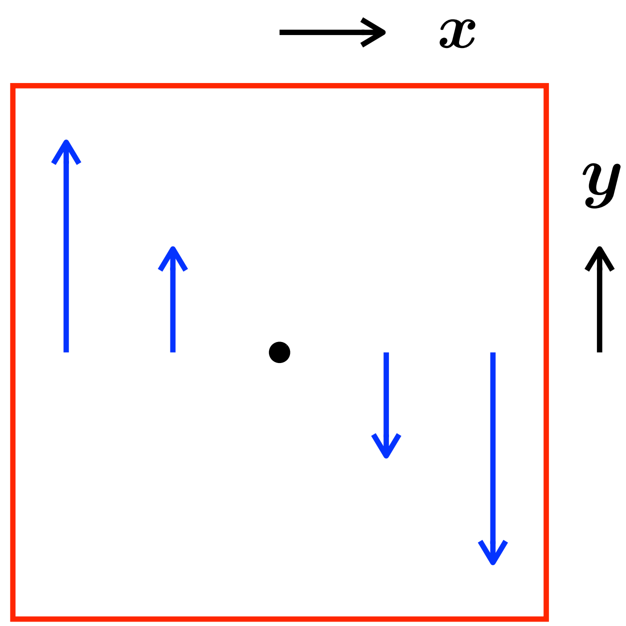
$$\boldsymbol{u} = \boldsymbol{u}_0 = -Sx\,\boldsymbol{e}_y \qquad \boldsymbol{B} = \boldsymbol{B}_0(t) \ \text{ with } \ \frac{\mathrm{d}\boldsymbol{B}_0}{\mathrm{d}t} = -SB_{x0}\,\boldsymbol{e}_y$$

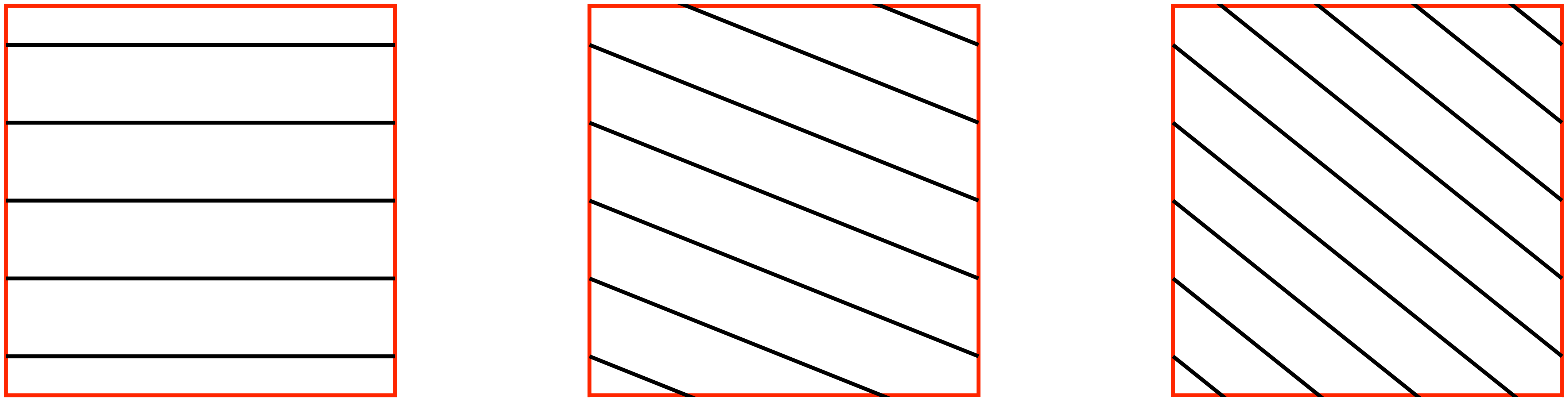
$$\Pi = \Pi_0 = \mathrm{cst} \qquad B_{x0} = \mathrm{cst} \qquad B_{y0} = \mathrm{cst} - SB_{x0}t \qquad B_{z0} = \mathrm{cst}$$

Magnetorotational instability

$$B_{x0} = \mathrm{cst} \qquad B_{y0} = \mathrm{cst} - SB_{x0}t \qquad B_{z0} = \mathrm{cst}$$

- Tilting / shearing of magnetic field:





Magnetorotational instability

- Perturbations in the form of shearing waves:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathrm{Re}\,\{\tilde{\boldsymbol{v}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + (\mu_0\rho)^{-1/2}\mathrm{Re}\left\{\tilde{\boldsymbol{b}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

$$\Pi = \Pi_0 + \rho\,\mathrm{Re}\left\{\tilde{\psi}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\} \qquad \text{with} \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = Sk_y\,\boldsymbol{e}_x$$

- Nonlinear terms vanish because

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{b} &= \mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\cdot\boldsymbol{\nabla}\,\mathrm{Re}\left[\tilde{\boldsymbol{b}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\ &= \mathrm{Re}\left[\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\mathrm{Re}\left[\mathrm{i}\tilde{\boldsymbol{b}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\ &= 0\end{aligned}$$

because $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$

and similarly for $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v},\ \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{v},\ \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}$

Magnetorotational instability

- Amplitude equations:

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y &= -\mathrm{i}k_x\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_x - \nu k^2\tilde{v}_x \\
\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x &= -\mathrm{i}k_y\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_y - \nu k^2\tilde{v}_y \\
\frac{\mathrm{d}\tilde{v}_z}{\mathrm{d}t} &= -\mathrm{i}k_z\tilde{\psi} + \mathrm{i}\omega_\mathrm{a}\tilde{b}_z - \nu k^2\tilde{v}_z \\
\frac{\mathrm{d}\tilde{b}_x}{\mathrm{d}t} &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_x - \eta k^2\tilde{b}_x \\
\frac{\mathrm{d}\tilde{b}_y}{\mathrm{d}t} &= -S\tilde{b}_x + \mathrm{i}\omega_\mathrm{a}\tilde{v}_y - \eta k^2\tilde{b}_y \\
\frac{\mathrm{d}\tilde{b}_z}{\mathrm{d}t} &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_z - \eta k^2\tilde{b}_z
\end{aligned}
$$

$$\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{b}} = 0$$

- Alfvén frequency $\omega_\mathrm{a} = \boldsymbol{k}\cdot\boldsymbol{v}_\mathrm{a} = (\mu_0\rho)^{-1/2}\boldsymbol{k}\cdot\boldsymbol{B}_0$

Magnetorotational instability

- Alfvén frequency is constant:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{k}\cdot\boldsymbol{B}_0) &= \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{B}_0 + \boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{B}_0}{\mathrm{d}t} \\
&= Sk_y\,\boldsymbol{e}_x\cdot\boldsymbol{B}_0 + \boldsymbol{k}\cdot(-SB_{x0}\,\boldsymbol{e}_y) \\
&= 0
\end{aligned}
$$

- Alfvén frequency measures the restoring effect of magnetic tension (amount of bending of field lines)

Magnetorotational instability

- General shearing waves require numerical solution
- Consider purely horizontal disturbances with a vertical wavevector:

$$k_x = k_y = 0 \qquad\qquad \tilde{v}_z = \tilde{b}_z = \tilde{\psi} = 0$$

- Amplitude equations have constant coefficients
- Solutions $\propto \mathrm{e}^{-\mathrm{i}\omega t}$, instability if $\mathrm{Im}(\omega) > 0$

$$\begin{aligned}
-\mathrm{i}\omega\tilde{v}_x - 2\Omega\tilde{v}_y &= \mathrm{i}\omega_\mathrm{a}\tilde{b}_x - \nu k^2\tilde{v}_x \\
-\mathrm{i}\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x &= \mathrm{i}\omega_\mathrm{a}\tilde{b}_y - \nu k^2\tilde{v}_y \\
-\mathrm{i}\omega\tilde{b}_x &= \mathrm{i}\omega_\mathrm{a}\tilde{v}_x - \eta k^2\tilde{b}_x \\
-\mathrm{i}\omega\tilde{b}_y &= -S\tilde{b}_x + \mathrm{i}\omega_\mathrm{a}\tilde{v}_y - \eta k^2\tilde{b}_y
\end{aligned}$$

- Set determinant to zero: magnetorotational dispersion relation

$$[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_\mathrm{a}^2]^2 - \underbrace{2\Omega(2\Omega - S)}_{\kappa^2}(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S\omega_\mathrm{a}^2 = 0$$

Magnetorotational instability

$$[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_\mathrm{a}^2]^2 - 2\Omega\underbrace{(2\Omega - S)}_{\kappa^2}(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S\omega_\mathrm{a}^2 = 0$$

- Case of zero magnetic field (or no bending of field, $\omega_\mathrm{a} = 0$):

$\omega = \pm\kappa - \mathrm{i}\nu k^2$ (epicyclic oscillation with viscous damping)

Magnetorotational instability

$$[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_{\mathrm{a}}^2]^2 - \underbrace{2\Omega(2\Omega - S)}_{\kappa^2}(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S \omega_{\mathrm{a}}^2 = 0$$

- Case of ideal MHD ($\nu = \eta = 0$):

$$\omega^4 - (2\omega_{\mathrm{a}}^2 + \kappa^2)\omega^2 + \omega_{\mathrm{a}}^2(\omega_{\mathrm{a}}^2 - 2\Omega S) = 0$$

$$\Rightarrow \ \omega^2 = \omega_{\mathrm{a}}^2 + \frac{1}{2}\kappa^2 \left[1 \pm \left(1 + \frac{16\omega_{\mathrm{a}}^2\Omega^2}{\kappa^4} \right)^{1/2} \right]$$

- Assume that $\kappa^2 > 0$, otherwise system is hydrodynamically unstable
- Both roots for ω^2 are real and at least one is positive
- Instability occurs if and only if product of roots < 0 , i.e.

$$0 < \omega_{\mathrm{a}}^2 < 2\Omega S$$

(Chandrasekhar’s criterion for “magnetorotational instability / MRI”)
(Velikhov 1959; Chandrasekhar 1960; ... ; Balbus & Hawley 1991)

Magnetorotational instability

- Unstable root:

$$\omega^2 = \omega_{\mathrm{a}}^2 + \frac{1}{2}\kappa^2 \left[1 - \left(1 + \frac{16\omega_{\mathrm{a}}^2\Omega^2}{\kappa^4} \right)^{1/2} \right]$$

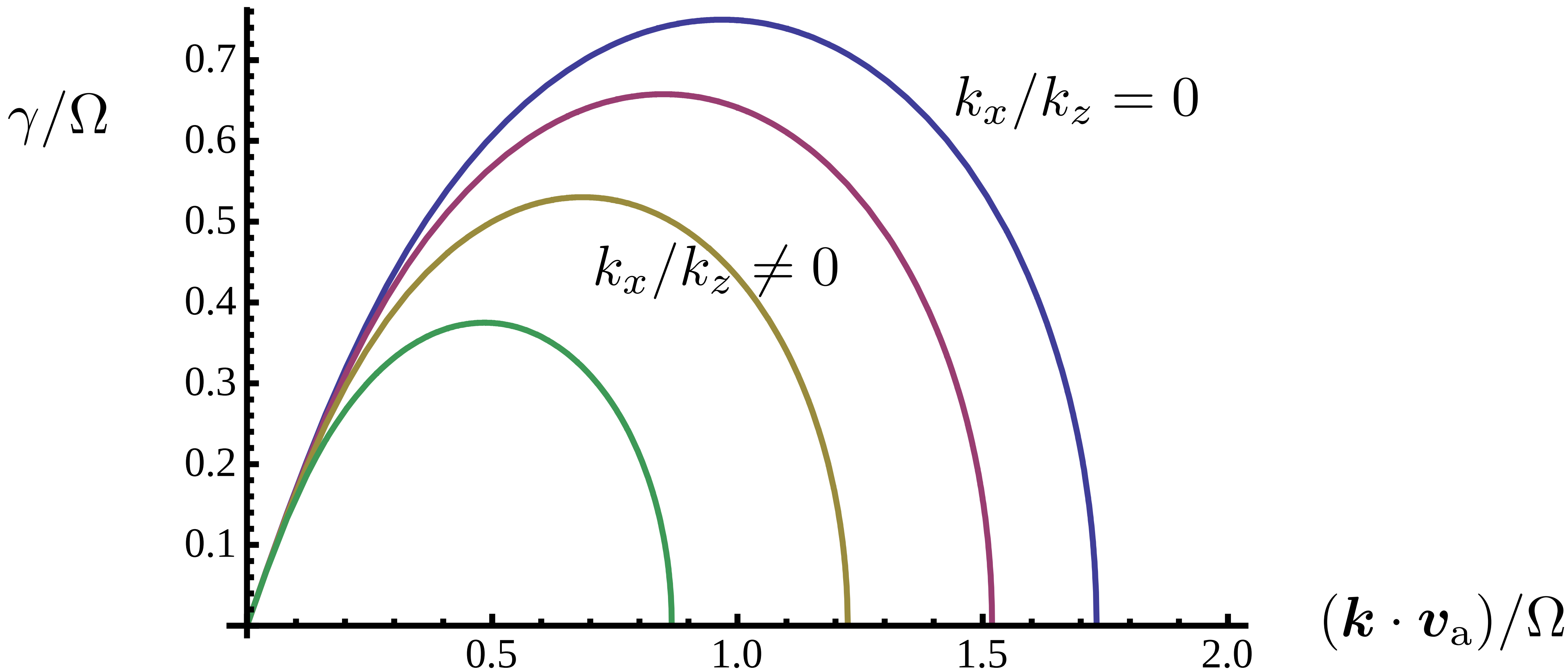
- Maximize growth rate with respect to k :

$$0 = \frac{\partial\omega^2}{\partial\omega_{\mathrm{a}}^2} = 1 - \frac{4\Omega^2}{\kappa^2}\left(1 + \frac{16\omega_{\mathrm{a}}^2\Omega^2}{\kappa^4} \right)^{-1/2} \qquad \Rightarrow \quad \omega_{\mathrm{a}}^2 = \Omega^2 - \frac{\kappa^4}{16\Omega^2}$$

$$\Rightarrow \quad (\omega^2)_{\min} = -\frac{S^2}{4}$$ so maximum growth rate is $\dfrac{S}{2}$

- Keplerian disc: energy grows by $\exp(3\pi) \approx 12392$ per orbit
- Optimal wavelength $2\pi\sqrt{\dfrac{16}{15}}\dfrac{v_{\mathrm{a}z}}{\Omega} \propto B_z$

Magnetorotational instability



Magnetorotational instability

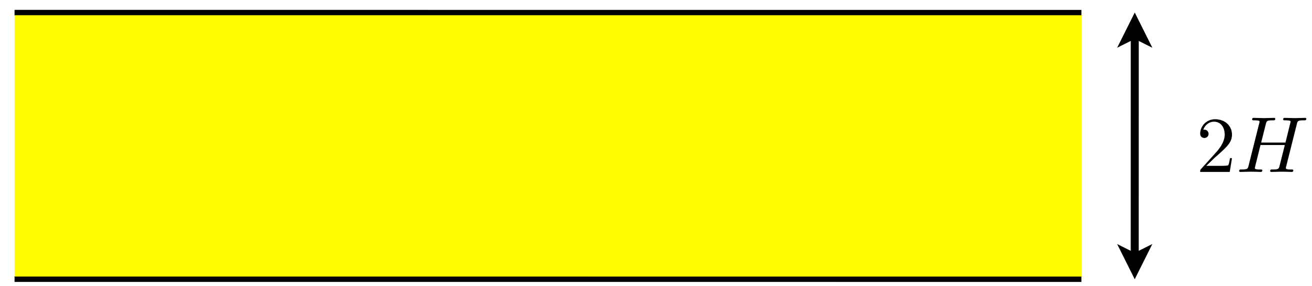
- As $B_z \to 0$ diffusion becomes more important
- Non-ideal MHD: if $\nu = \eta$ (for simplicity) then

$$\omega = \omega_{\mathrm{ideal}} - \mathrm{i}\eta k^2 \qquad \text{(reduces growth rate)}$$

- If k can take any value then instability persists for small k

Magnetorotational instability

- Effect of vertical boundaries:



- Suppose $k = \dfrac{n\pi}{2H}, \quad n \in \mathbf{Z}$
- $n = 0$ mode gives no instability, so consider $n = 1$:
- Instability in ideal MHD when

$$0 < \omega_{\mathrm{a}}^2 < 2\Omega S \qquad \Rightarrow \; 0 < v_{\mathrm{a}} < \frac{2\sqrt{3}}{\pi} H\Omega \qquad \text{(Keplerian)}$$

- Diffusive damping rate of $n = 1$ mode $= \eta(\pi/2H)^2$
- Ideal growth rate $\sim \omega_{\mathrm{a}} = v_{\mathrm{a}}(\pi/2H)$
- Instability occurs for an intermediate range of field strengths,

roughly $\dfrac{\eta}{H} \lesssim v_{\mathrm{a}} \lesssim c_{\mathrm{s}}$

Magnetorotational instability

- Summary:
  - Hydrodynamic instability when

    $$2\Omega(2\Omega - S) < 0 \qquad \text{(Rayleigh)}$$

  - Magnetohydrodynamic instability (weak field, ideal MHD) when

    $$-2\Omega S < 0 \qquad \text{(Chandrasekhar)}$$

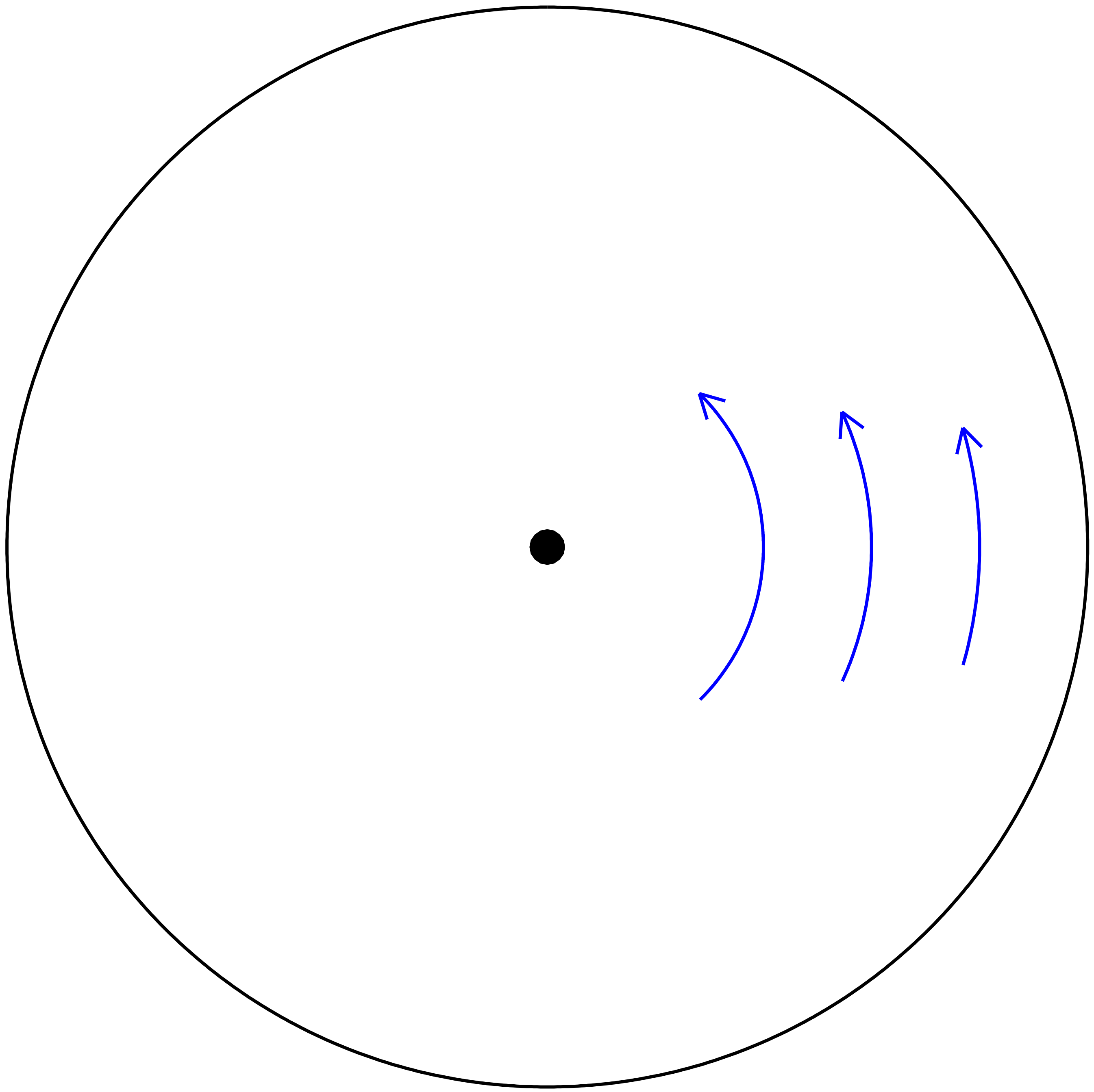
  - Paradox of $|\boldsymbol{B}| \to 0$ resolved by going to non-ideal MHD
  - In cylindrical geometry:

    $$\frac{\mathrm{d}}{\mathrm{d}r}(r^2|\Omega|) < 0 \quad \text{(Rayleigh)} \quad \text{versus} \quad \frac{\mathrm{d}}{\mathrm{d}r}|\Omega| < 0 \quad \text{(MRI)}$$

  - Usual situation in astrophysical discs:
    Rayleigh-stable but MRI-unstable

Magnetorotational instability

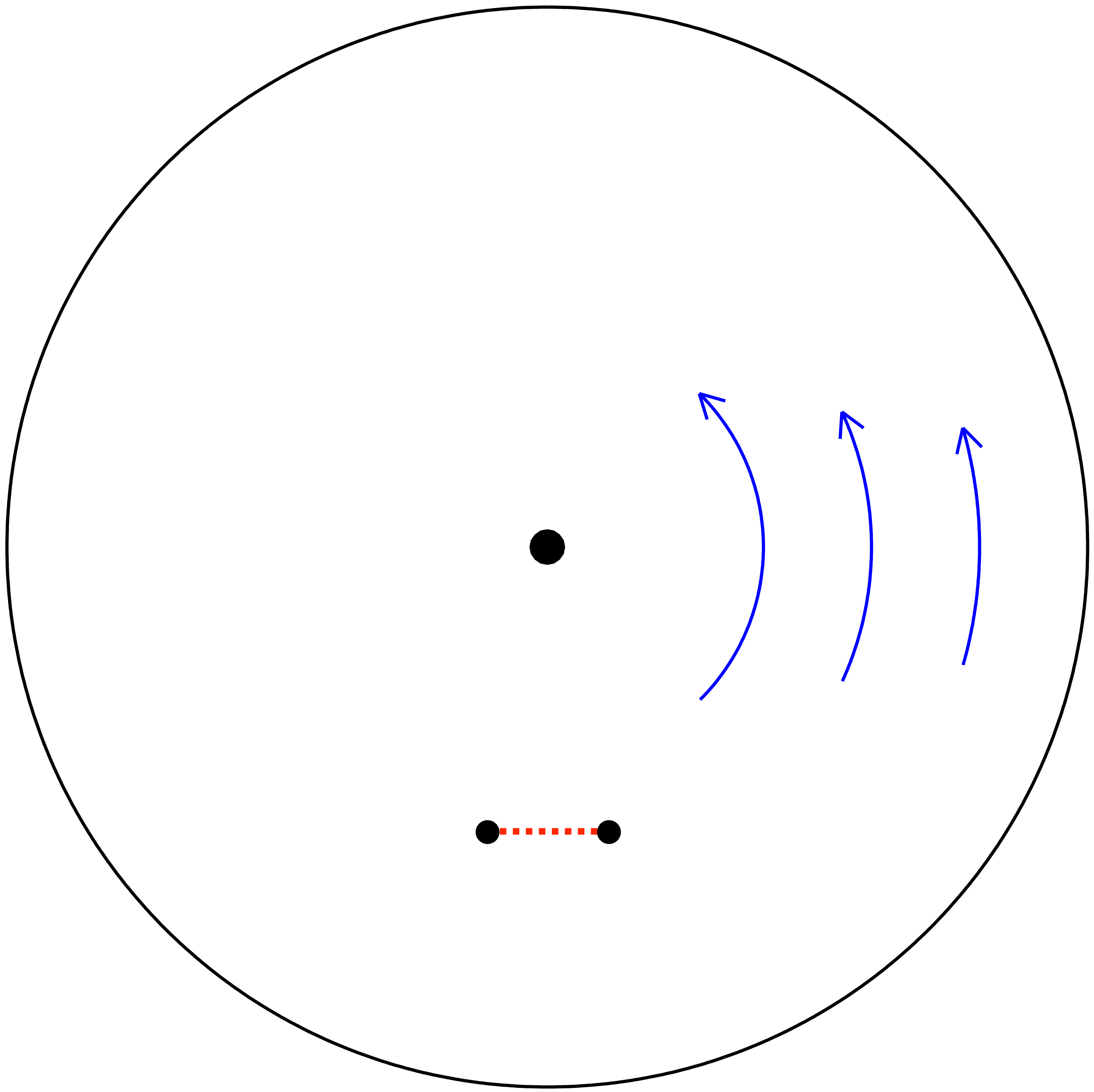
- Physical interpretation / mechanical analogy:



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

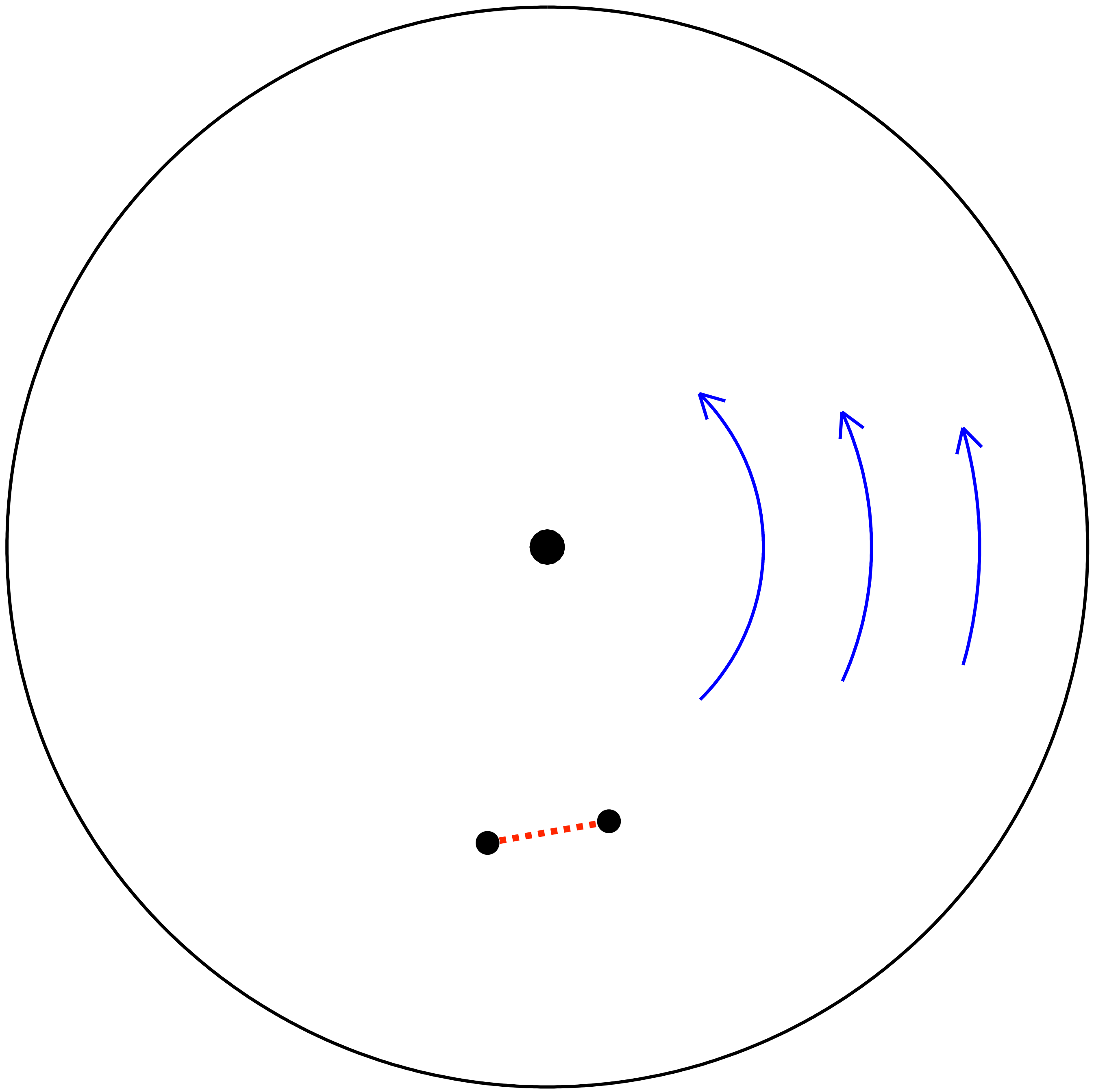
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

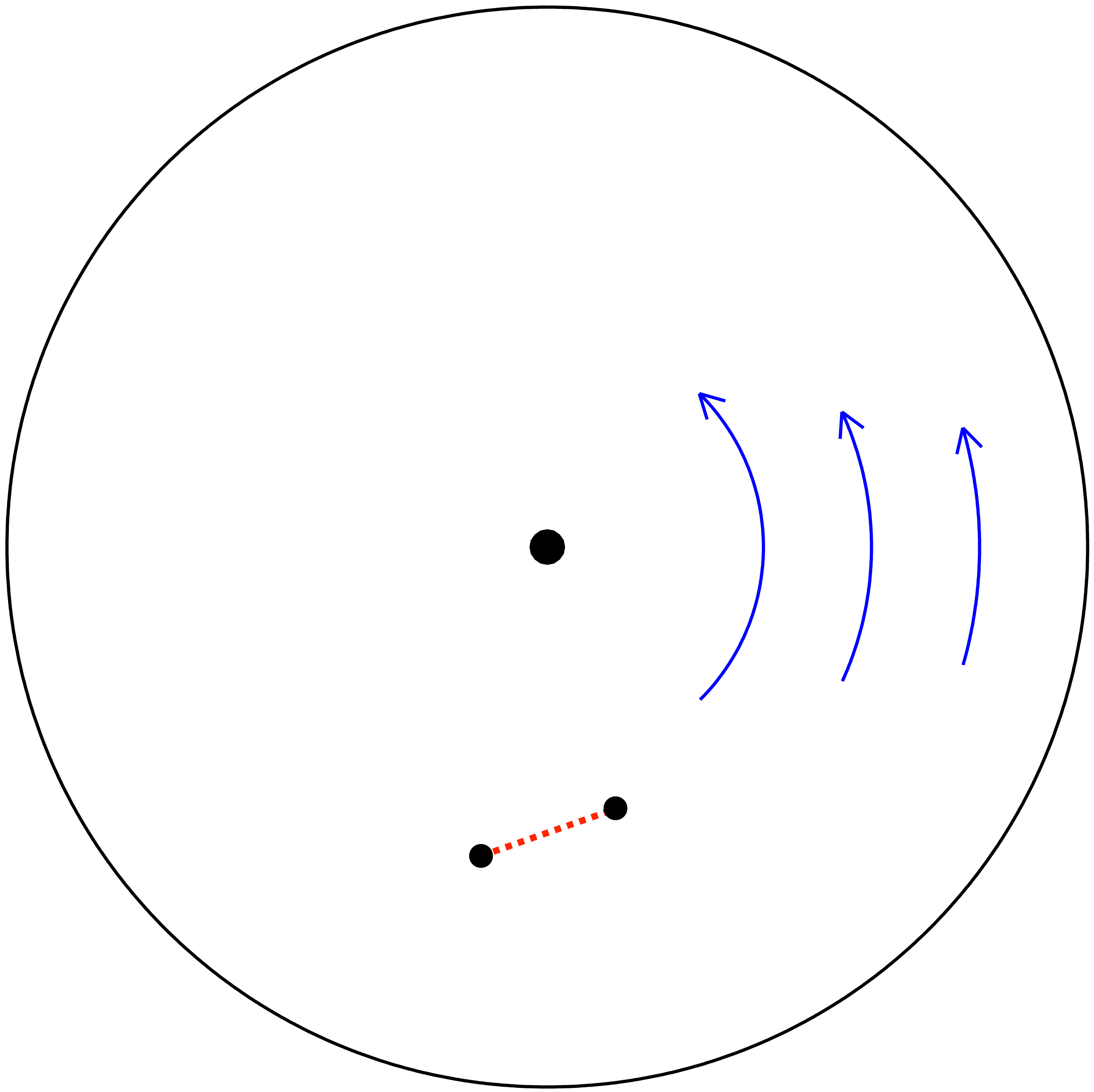
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

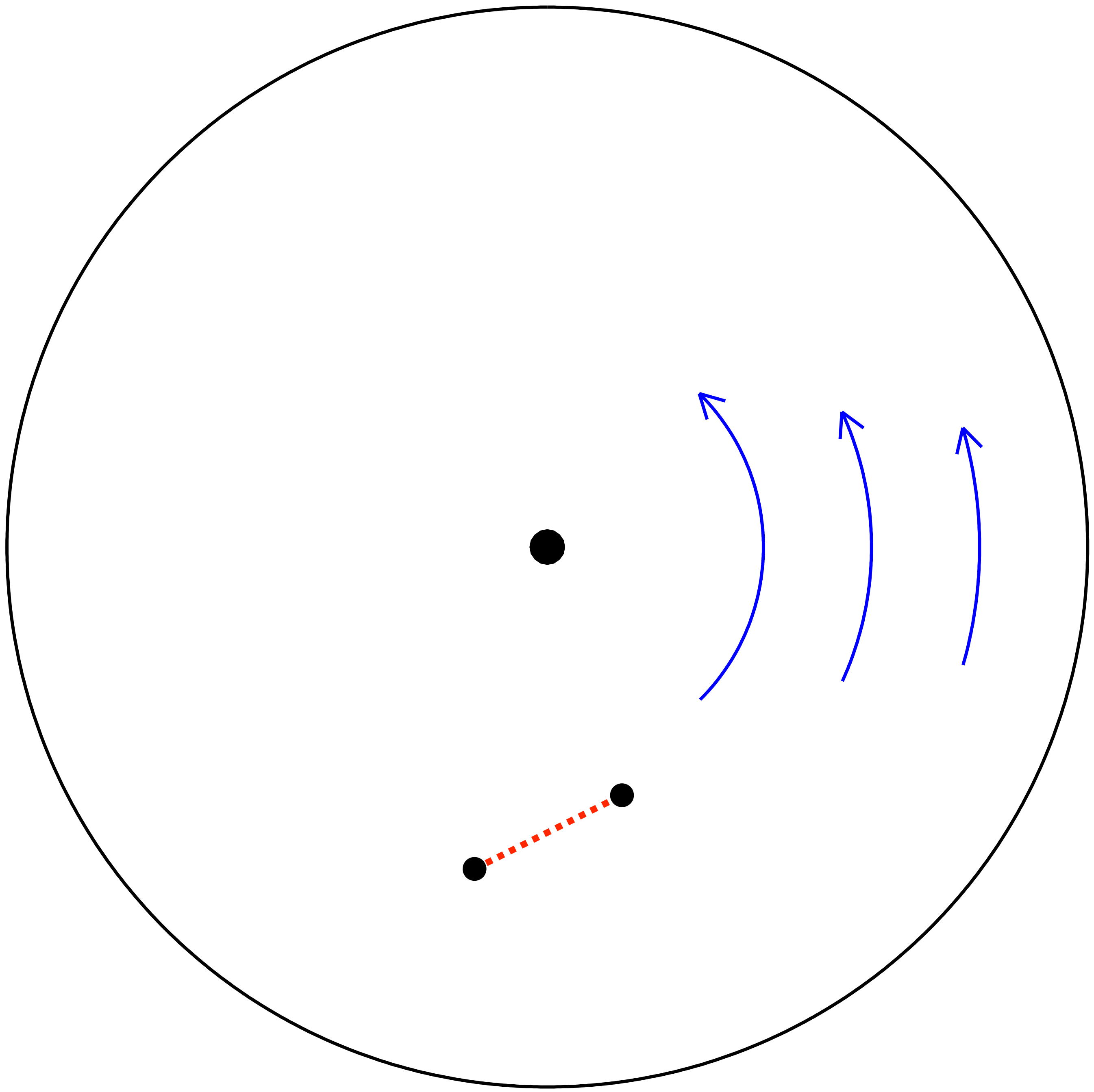
Magnetorotational instability



$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

Magnetorotational instability

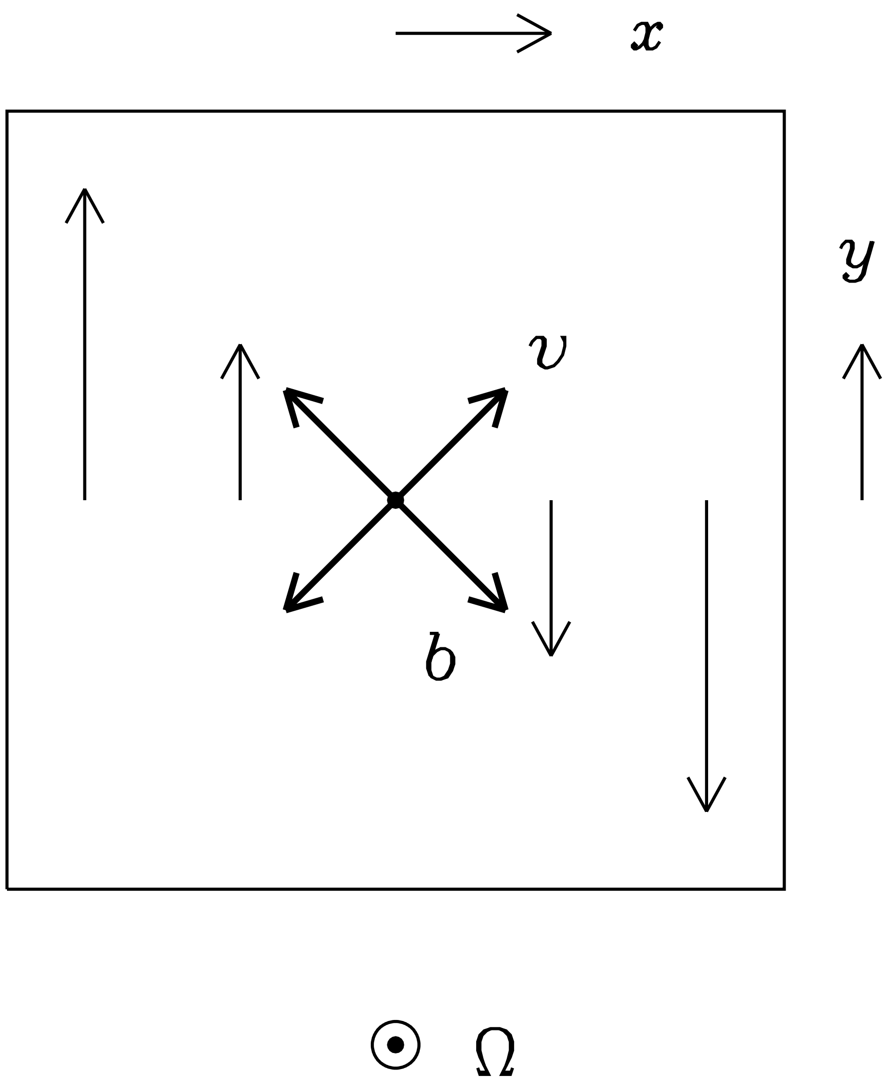


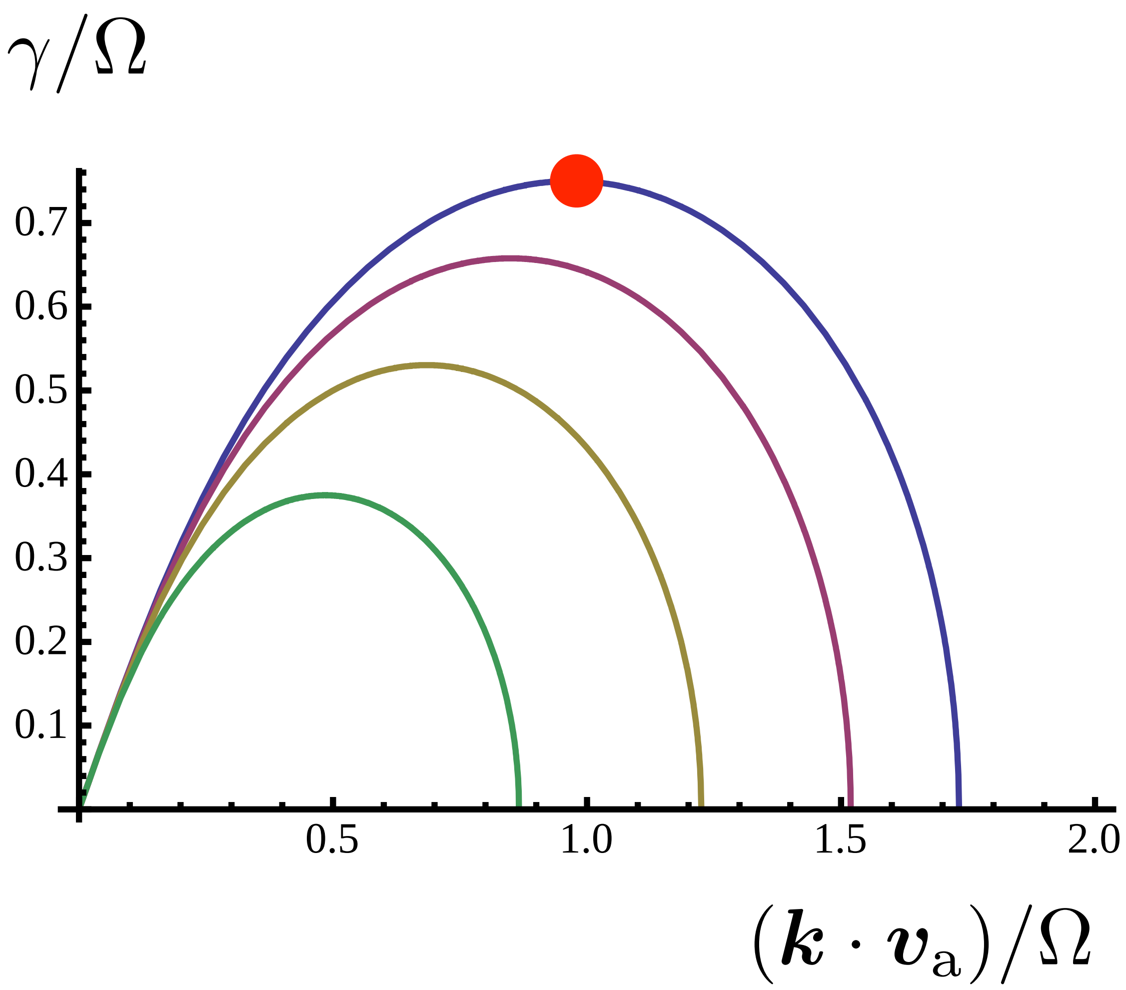
$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0$$

$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0$$

Magnetorotational instability

Optimal “channel mode”:





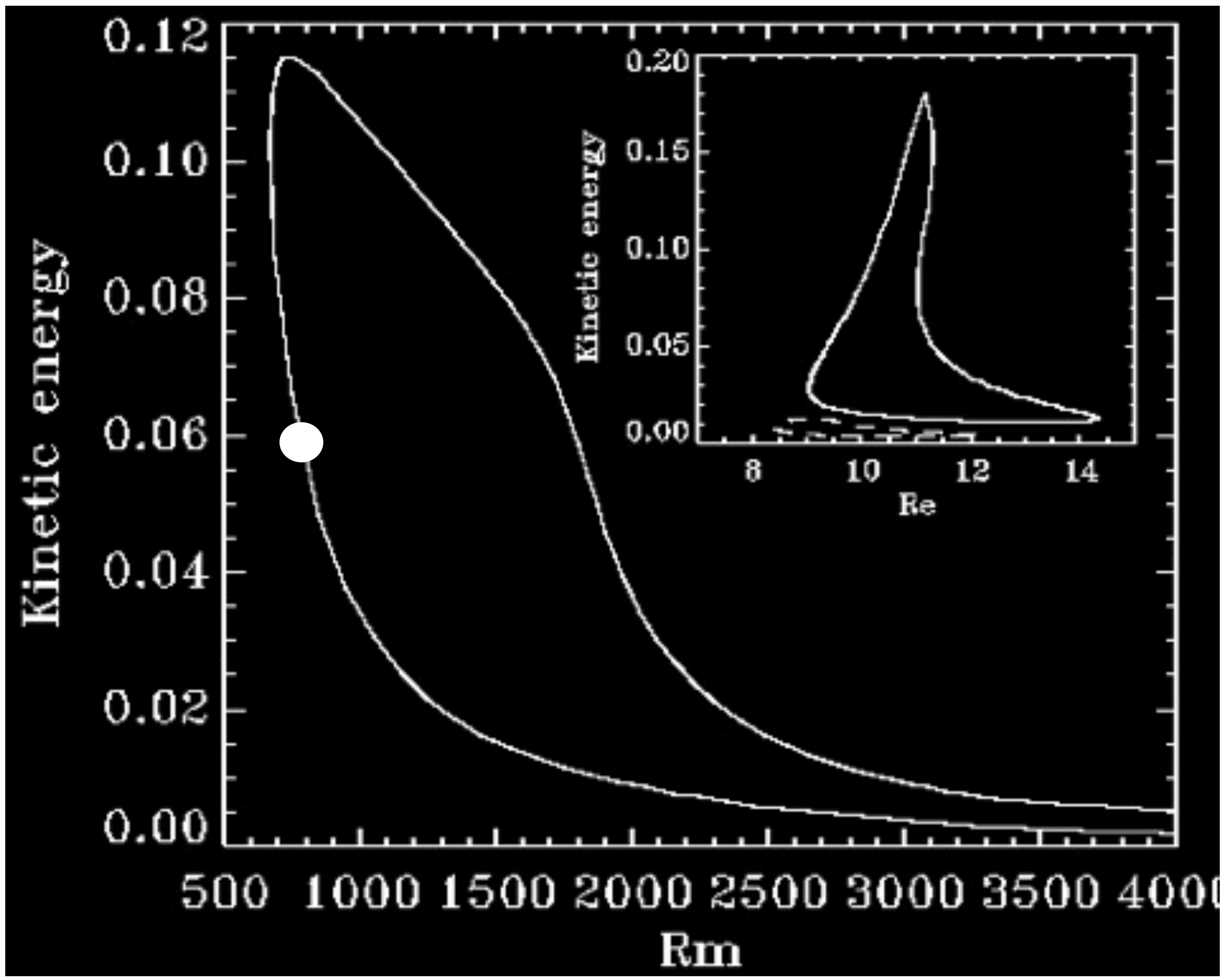
Magnetorotational instability

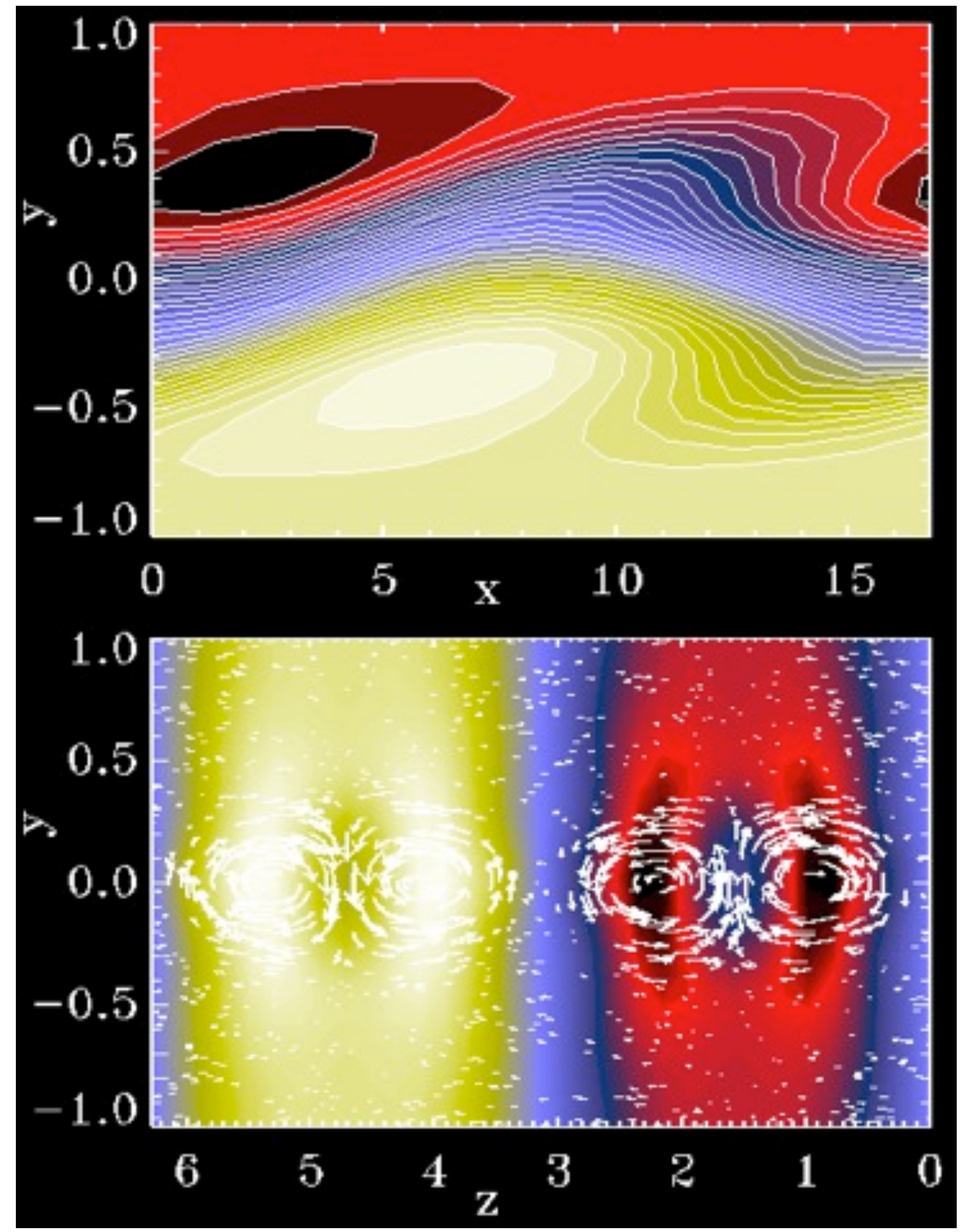
- Nonlinear outcome:
  - With imposed magnetic field: sustained MHD turbulence (intensity depends on imposed magnetic field)
  - Without imposed magnetic field: nonlinear dynamo?

$$\boxed{\boldsymbol{B}} \;\rightleftarrows\; \boxed{\boldsymbol{u}}$$

Magnetorotational instability

- Steady dynamo solutions found by numerical continuation / Newton iteration (Rincon et al. 2007)





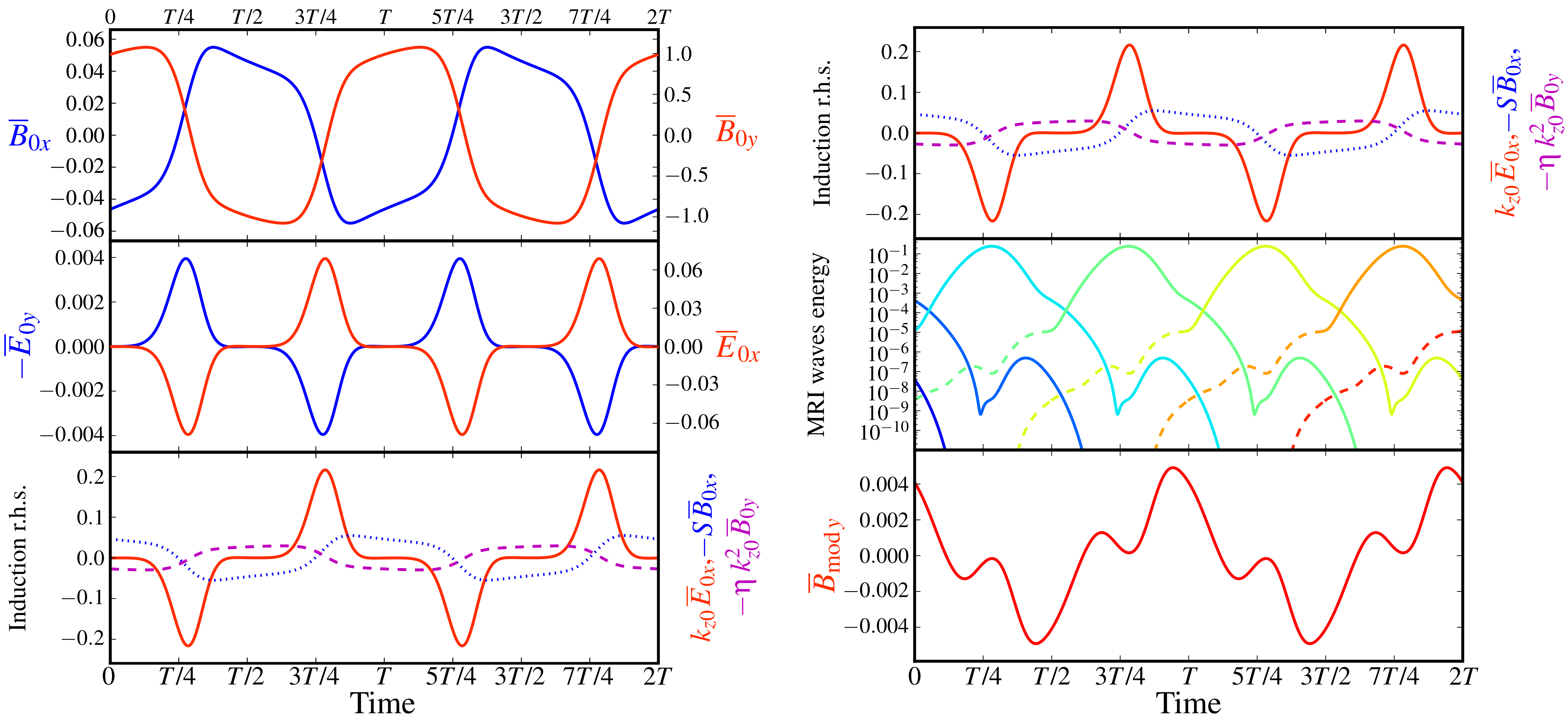
- dominant toroidal field
- Rm > 680, but low Re only !
- verified by spectral DNS

magnetic field

Rincon et al, 2007: Phys. Rev. Lett., 98, 254502

Magnetorotational instability

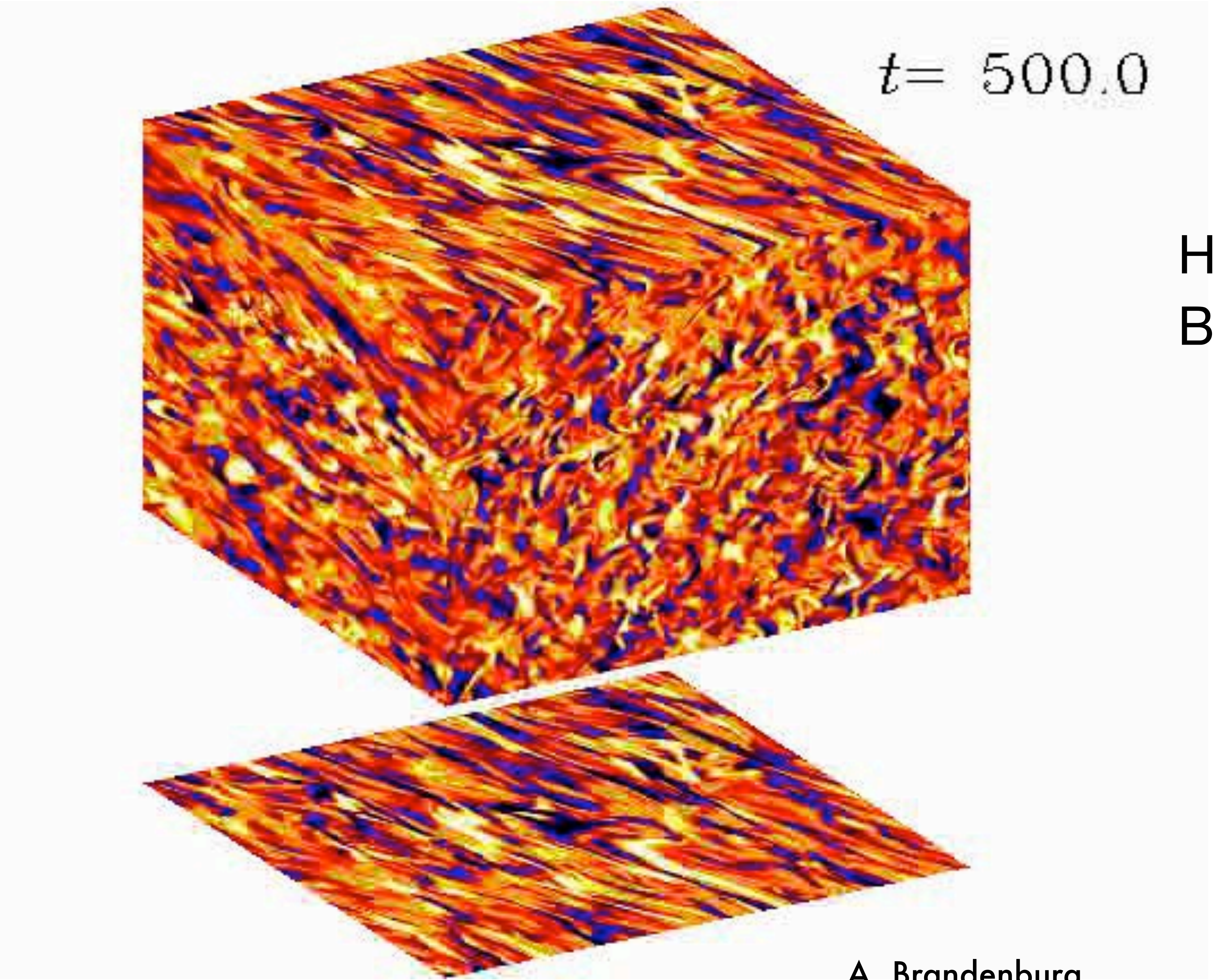
- Periodic dynamo solutions in shearing box (Herault et al. 2011)



Herault et al. 2011:
Phys. Rev. E, 84, 036321

Magnetorotational instability

- Sustained MHD turbulence



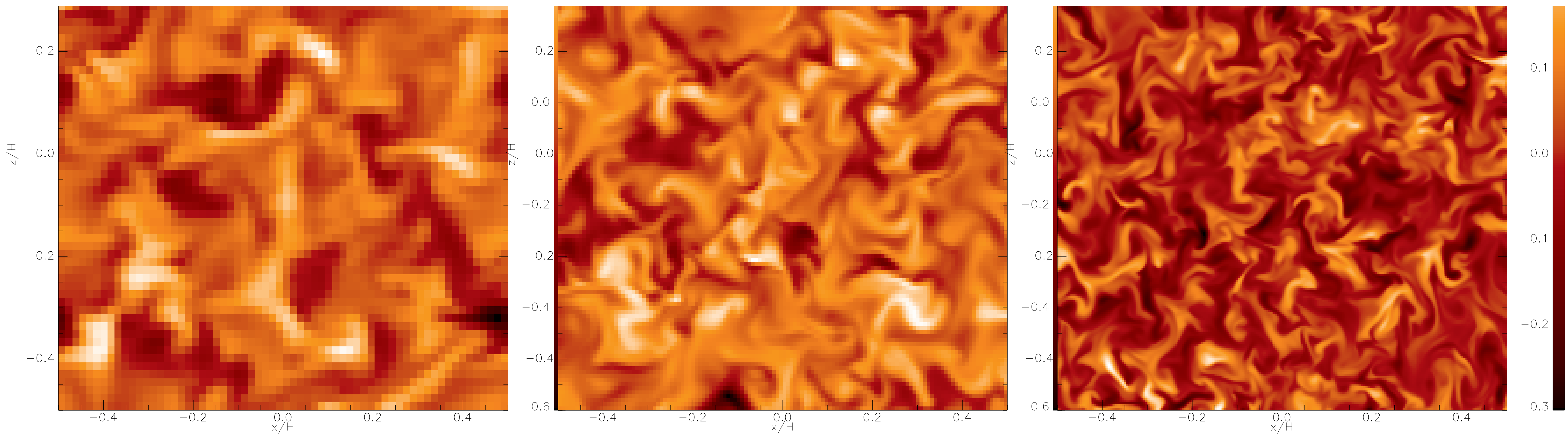
A. Brandenburg

Hawley et al. (1995) ...
Brandenburg et al. (1995) ...

http://www.nordita.org/software/pencil-code/movies/disc/BH256_3D_mean_Bz=0b1bz.mpg

Magnetorotational instability

- Controversy over saturation and turbulent transport (Fromang & Papaloizou 2007)

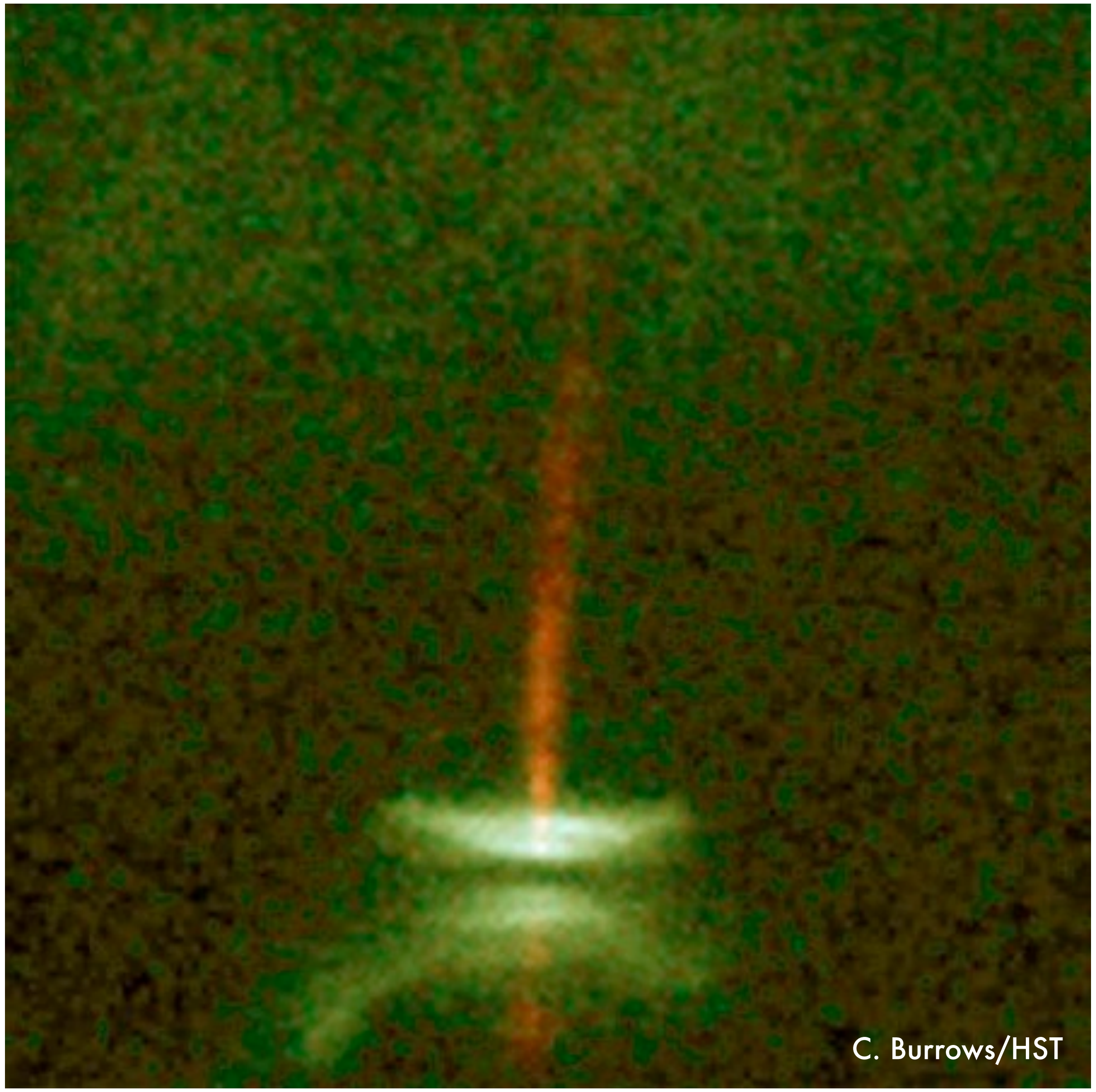


64.100.64	128.200.128	256.400.256
α = 0.0040	α = 0.0021	α = 0.0011

Fromang & Papaloizou, 2007: A&A, 476, 3, 1113

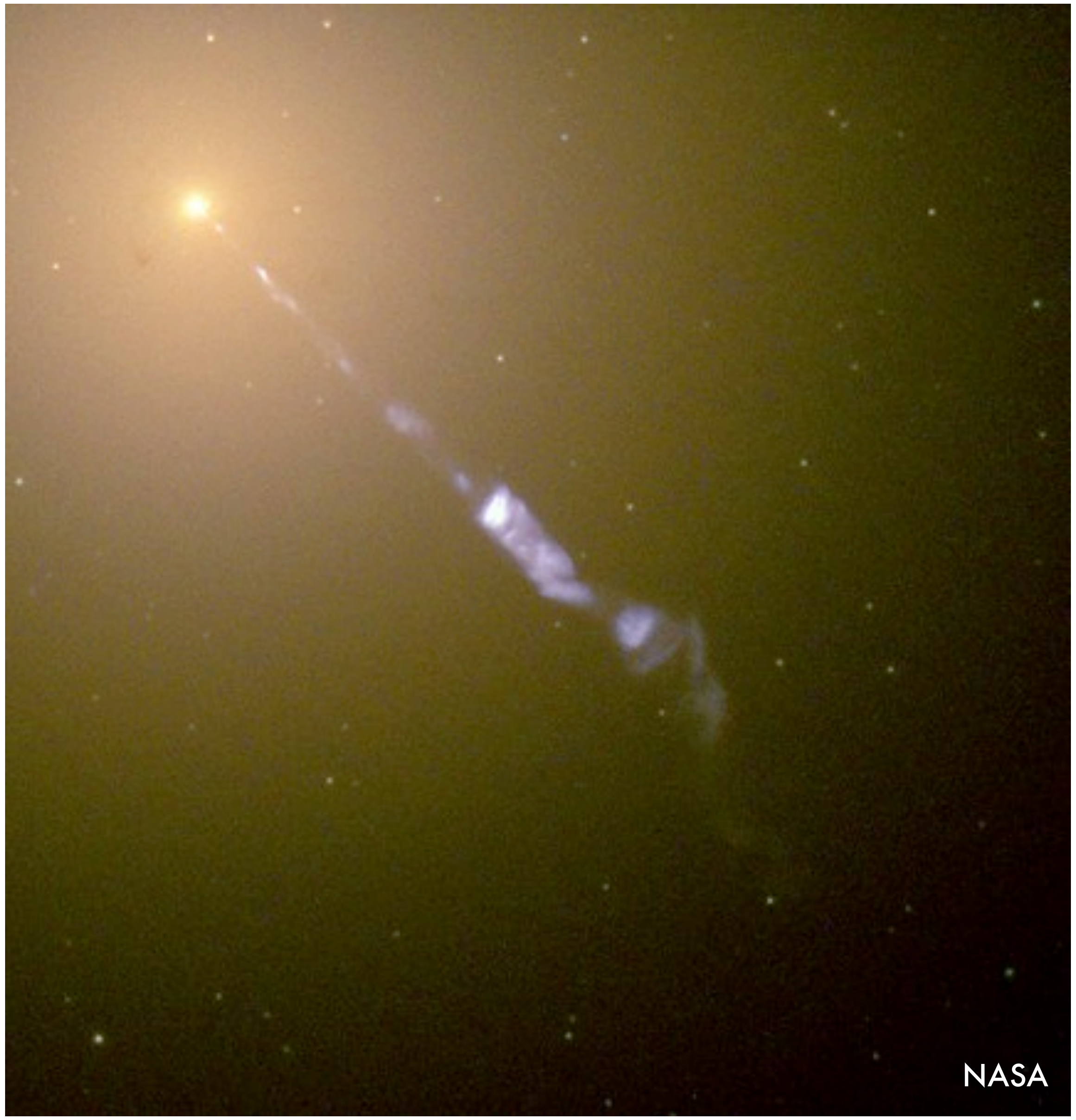
Jet launching

- Jets and winds are commonly observed to be emitted by systems containing discs



HH30 (young star)

HubbleSite - NewsCenter - A Cosmic Searchlight (07/06/2000) - Introduction
http://hubblesite.org/newscenter/archive/releases/2000/20



M87 (giant elliptical galaxy)

HubbleSite - Picture Album: Reddish Jet of Gas Emanates From Forming Star HH-30
http://hubblesite.org/gallery/album/entire/pr1995024e/

Jet launching

- Develop theory of steady axisymmetric flows in ideal MHD
- Cylindrical polar coordinates (r, ϕ, z)
- Solutions independent of ϕ and t
- Representation of an axisymmetric magnetic field:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{r} \frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$$

- Introduce magnetic flux function $\psi(r, z)$:

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} , \qquad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

- Related to vector potential ($\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$) through $\psi = r A_\phi$
- Magnetic flux through the circle $r = r_0$, $z = z_0$ is

$$\int_0^{r_0} B_z(r, z_0)\, 2\pi r \, \mathrm{d}r = 2\pi \psi(r_0, z_0) \quad (+\mathrm{cst})$$

- Since $\boldsymbol{B} \cdot \boldsymbol{\nabla}\psi = 0$, ψ labels magnetic field lines (or surfaces)

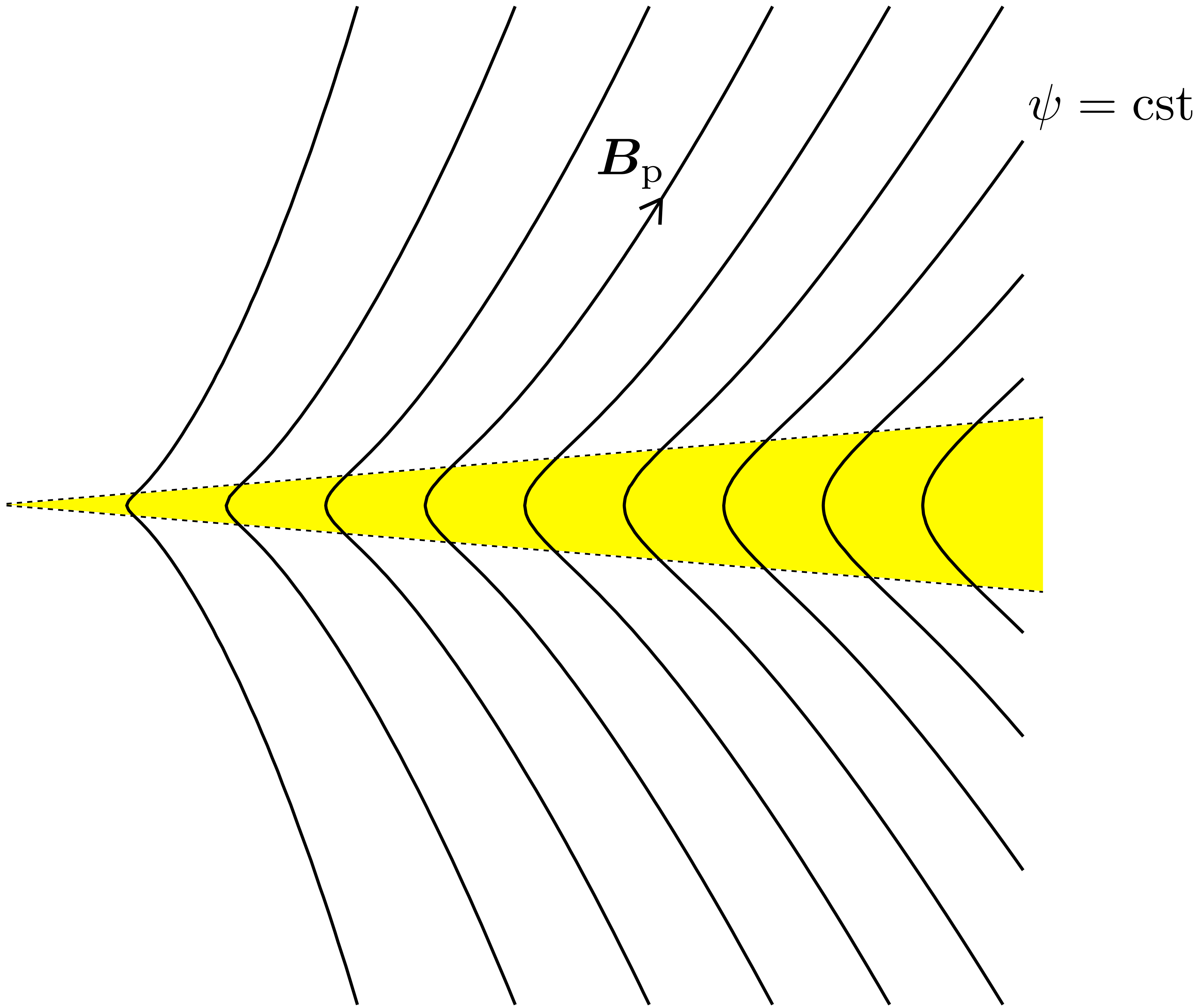
Jet launching

- Thus

$$\boldsymbol{B} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi + B_\phi\, \boldsymbol{e}_\phi = \underbrace{\left[-\frac{1}{R}\boldsymbol{e}_\phi \times \boldsymbol{\nabla}\psi \right]}_{\substack{\text{poloidal part} \\ \text{(meridional)} \\ \boldsymbol{B}_\mathrm{p}}} + \underbrace{\left[B_\phi\, \boldsymbol{e}_\phi \right]}_{\substack{\text{toroidal part} \\ \text{(azimuthal)} \\ \boldsymbol{B}_\mathrm{t}}}$$

- Can also write $\boldsymbol{B}_\mathrm{p} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi$
- Note that $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{B}_\mathrm{p} = 0$
- Can similarly write $\boldsymbol{u} = \boldsymbol{u}_\mathrm{p} + u_\phi\, \boldsymbol{e}_\phi$

Jet launching



Jet launching

- Steady induction equation in ideal MHD:

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0} \qquad \Rightarrow \ \boldsymbol{u} \times \boldsymbol{B} = -\boldsymbol{E} = \boldsymbol{\nabla}\Phi_{\mathrm{e}}$$

electrostatic potential

- But $\boldsymbol{u} \times \boldsymbol{B} = (\boldsymbol{u}_{\mathrm{p}} + u_\phi \, \boldsymbol{e}_\phi) \times (\boldsymbol{B}_{\mathrm{p}} + B_\phi \, \boldsymbol{e}_\phi)$

$$= \underbrace{\Big[\boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_{\mathrm{p}} - B_\phi \boldsymbol{u}_{\mathrm{p}})\Big]}_{\text{poloidal part}} + \underbrace{\Big[\boldsymbol{u}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}}\Big]}_{\text{toroidal part}}$$

- For an axisymmetric solution with $\partial\Phi_{\mathrm{e}}/\partial\phi = 0$:

$$\boldsymbol{u}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}} = \boldsymbol{0} \qquad \Rightarrow \ \boldsymbol{u}_{\mathrm{p}} \parallel \boldsymbol{B}_{\mathrm{p}}$$

- Introduce mass loading k (ratio of mass flux to magnetic flux):

$$\rho \boldsymbol{u}_{\mathrm{p}} = k \boldsymbol{B}_{\mathrm{p}}$$

Jet launching

- Steady equation of mass conservation:

$$0 = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_{\mathrm{p}}) = \boldsymbol{\nabla} \cdot (k \boldsymbol{B}_{\mathrm{p}}) = \boldsymbol{B}_{\mathrm{p}} \cdot \boldsymbol{\nabla} k$$

- Therefore $k = k(\psi)$ is constant on each magnetic surface
- We now have

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_{\mathrm{p}} - B_\phi \boldsymbol{u}_{\mathrm{p}}) = \left(\frac{u_\phi}{r} - \frac{k B_\phi}{r\rho} \right) \boldsymbol{\nabla} \psi$$

- Take the curl:

$$\boldsymbol{0} = \boldsymbol{\nabla} \left(\frac{u_\phi}{r} - \frac{k B_\phi}{r\rho} \right) \times \boldsymbol{\nabla} \psi$$

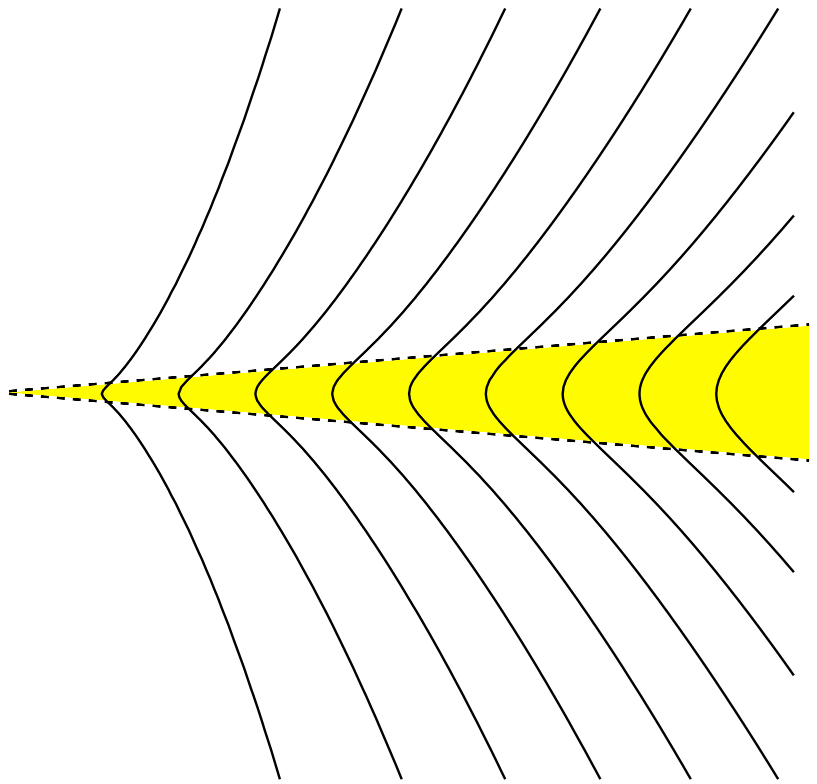
- Therefore

$$\frac{u_\phi}{r} - \frac{k B_\phi}{r\rho} = \omega(\psi)$$ angular velocity of magnetic surface

- Total velocity field $\boldsymbol{u} = \dfrac{k\boldsymbol{B}}{\rho} + r\omega\, \boldsymbol{e}_\phi$

Jet launching

- Total velocity field $\boldsymbol{u} = \dfrac{k\boldsymbol{B}}{\rho} + r\omega\, \boldsymbol{e}_\phi$
- Total velocity is parallel to total magnetic field in a frame rotating with the magnetic surface
- Fluid is constrained like a bead on a rotating wire



- Steady thermal energy equation:

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} s = 0$$

$$\Rightarrow \quad \boldsymbol{B}_\mathrm{p} \cdot \boldsymbol{\nabla} s = 0$$

$$\Rightarrow \quad s = s(\psi)$$ another surface function

Jet launching

- Azimuthal component of equation of motion:

$$\rho\left(\boldsymbol{u}_{\mathrm{p}}\cdot\boldsymbol{\nabla}u_\phi+\frac{u_r u_\phi}{r}\right)=\frac{1}{\mu_0}\left(\boldsymbol{B}_{\mathrm{p}}\cdot\boldsymbol{\nabla}B_\phi+\frac{B_r B_\phi}{r}\right)$$

$$\frac{1}{r}\rho\boldsymbol{u}_{\mathrm{p}}\cdot\boldsymbol{\nabla}(ru_\phi)-\frac{1}{\mu_0 r}\boldsymbol{B}_{\mathrm{p}}\cdot\boldsymbol{\nabla}(rB_\phi)=0$$

$$\frac{1}{r}\boldsymbol{B}_{\mathrm{p}}\cdot\boldsymbol{\nabla}\left(kru_\phi-\frac{rB_\phi}{\mu_0}\right)=0$$

$$ru_\phi=\frac{rB_\phi}{\mu_0 k}+\ell(\psi)$$

angular momentum invariant of magnetic surface

- $\ell(\psi)$ is the angular momentum removed per unit mass in the outflow, some of which is carried by the magnetic field

Jet launching

- Alfvén surface:
- Define the poloidal Alfvén number (cf. Mach number) $A = \dfrac{|\boldsymbol{u}_\mathrm{p}|}{|\boldsymbol{v}_\mathrm{ap}|}$
- Then

$$A^2 = \frac{\mu_0 \rho u_\mathrm{p}^2}{B_\mathrm{p}^2} = \frac{\mu_0 k^2}{\rho} \qquad \Rightarrow A \propto \rho^{-1/2}$$

on each magnetic surface

- Consider the two equations

$$\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} = \omega(\psi)$$

$$ru_\phi = \frac{rB_\phi}{\mu_0 k} + \ell(\psi)$$

- Eliminate B_ϕ :

$$u_\phi = \frac{r^2\omega - A^2\ell}{r(1-A^2)} = \left(\frac{1}{1-A^2}\right) r\omega + \left(\frac{A^2}{A^2-1}\right)\frac{\ell}{r}$$

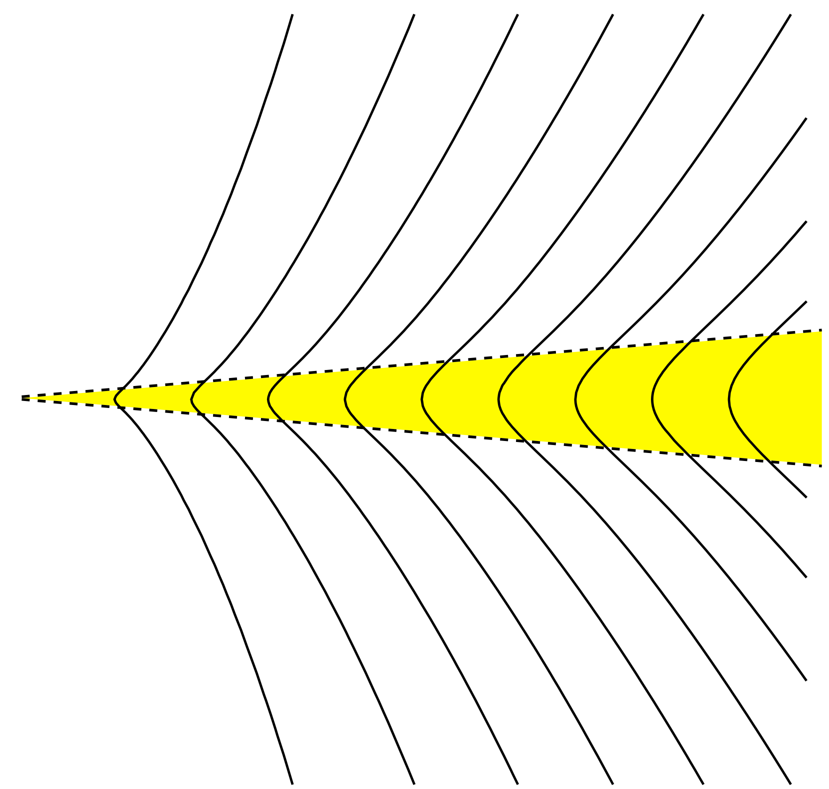
Jet launching

$$u_\phi = \frac{r^2\omega - A^2\ell}{r(1-A^2)} = \left(\frac{1}{1-A^2}\right) r\omega + \left(\frac{A^2}{A^2-1}\right)\frac{\ell}{r}$$

- For $A \ll 1$: $u_\phi \approx r\omega$ (fluid corotates with magnetic surface)
- For $A \gg 1$: $u_\phi \approx \dfrac{\ell}{r}$ (fluid conserves specific angular momentum)
- Alfvén point $(r = r_{\mathrm{a}}(\psi))$ where $A = 1$
- Alfvén surface is locus of all such points for different ψ
- To avoid a singularity, require $\ell = r_{\mathrm{a}}^2\omega$

Jet launching

- Typically, flow starts in high-density material where $A \ll 1$
- Identify ω as the angular velocity Ω_0 of the “footpoint” at r_0
- Successful outflow accelerates outwards and achieves $A > 1$



- If mass is lost at a rate $\dot{M}$ in the outflow, angular momentum is lost at a rate $\dot{M}\ell = \dot{M} r_{\rm a}^2 \Omega_0$
- In contrast, in a purely hydrodynamic outflow, angular momentum is lost at a rate $\dot{M} r_0^2 \Omega_0$
- This effect is the magnetic lever arm
- Loss of angular momentum by a magnetized outflow is called magnetic braking

Jet launching

- Total energy equation for steady flow:

$$\boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+w)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]=0$$

- Now

$$\boldsymbol{u}=\frac{k\boldsymbol{B}}{\rho}+r\omega\,\boldsymbol{e}_\phi$$

$$\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}=-r\omega\,\boldsymbol{e}_\phi\times\boldsymbol{B}=-r\omega\,\boldsymbol{e}_\phi\times\boldsymbol{B}_\mathrm{p}$$ (purely poloidal)

$$(\boldsymbol{E}\times\boldsymbol{B})_\mathrm{p}=\boldsymbol{E}\times(B_\phi\,\boldsymbol{e}_\phi)=-r\omega B_\phi\boldsymbol{B}_\mathrm{p}$$

$$\boldsymbol{\nabla}\cdot\left[k\boldsymbol{B}_\mathrm{p}(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+w)-\frac{r\omega B_\phi}{\mu_0}\boldsymbol{B}_\mathrm{p}\right]=0$$

$$\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\left[k\left(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+w-\frac{r\omega B_\phi}{\mu_0 k}\right)\right]=0$$

$$\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+w-\frac{r\omega B_\phi}{\mu_0 k}=\varepsilon(\psi)$$ energy invariant

Jet launching

- Alternative invariant:

$$\begin{aligned}\tilde{\varepsilon} &= \varepsilon - \ell\omega \\ &= \tfrac{1}{2}|\boldsymbol{u}|^2 + \Phi + w - \frac{r\omega B_\phi}{\mu_0 k} - \left(ru_\phi - \frac{rB_\phi}{\mu_0 k}\right)\omega \\ &= \tfrac{1}{2}|\boldsymbol{u}|^2 + \Phi + w - ru_\phi\omega \\ &= \tfrac{1}{2}|\boldsymbol{u}_\mathrm{p}|^2 + \tfrac{1}{2}(u_\phi - r\omega)^2 + \Phi^\mathrm{cg} + w\end{aligned}$$

- Centrifugal-gravitational potential $\Phi^\mathrm{cg} = \Phi - \frac{1}{2}\omega^2 r^2$
- Identify $\tilde{\varepsilon}$ as the Bernoulli function in a frame that corotates with the magnetic surface
- Magnetic field does no work in this frame because $\boldsymbol{B} \parallel \boldsymbol{u}$ and so $(\boldsymbol{J} \times \boldsymbol{B}) \perp \boldsymbol{u}$

Jet launching

- Summary:
- We have integrated almost all the equations of ideal MHD
- Algebraic equations on each magnetic surface
- If $\boldsymbol{B}_{\mathrm{p}}$ (or ψ) is specified in advance, we can solve these to determine the flow, subject also to
  - initial conditions at the source of the outflow
  - smooth passage through critical points where flow speed matches speeds of slow and fast magnetoacoustic waves
- One remaining equation (component of equation of motion $\perp \boldsymbol{B}_{\mathrm{p}}$) is a very complicated nonlinear PDE that determines ψ
- “Transfield” or “Grad-Shafranov” equation (too difficult to consider here)

Jet launching

- Acceleration from the surface of a disc:
- Assume flow starts from rest ($A \ll 1$) from a Keplerian disc
- In the sub-Alfvénic region, $\tilde{\varepsilon} \approx \frac{1}{2}|\boldsymbol{u}_\mathrm{p}|^2 + \Phi^\mathrm{cg} + w$
- Can an outflow be accelerated mechanically (rather than thermally)? i.e. because Φ^cg (rather than w) decreases along the field line?
- Consider field line with footpoint radius r_0 and angular velocity

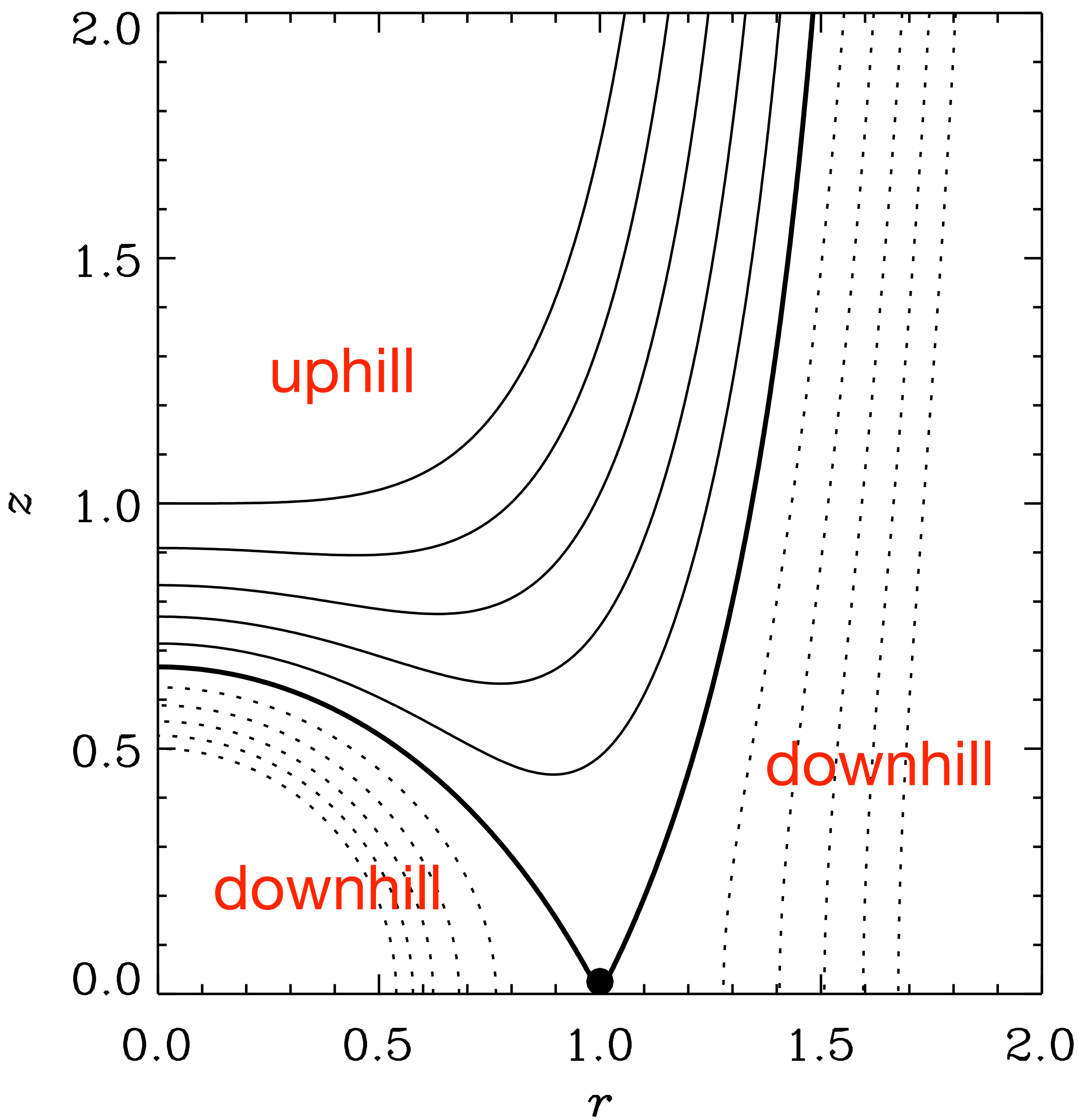
$$\omega = \Omega_0 = \left(\frac{GM}{r_0^3}\right)^{1/2}$$

- Then

$$\Phi^\mathrm{cg} = -GM(r^2+z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2$$

Jet launching

$$\Phi^{\rm cg} = -GM(r^2 + z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2$$



units:
$r_0 = 1$

Jet launching

$$\Phi^{\rm cg} = -GM(r^2+z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2$$

- Equation of critical equipotential:

$$(r^2+z^2)^{-1/2} + \frac{r^2}{2} = \frac{3}{2}$$

units: $r_0 = 1$

$$z^2 = \frac{(2-r)(r-1)^2(r+1)^2(r+2)}{(3-r^2)^2}$$

- Close to footpoint:

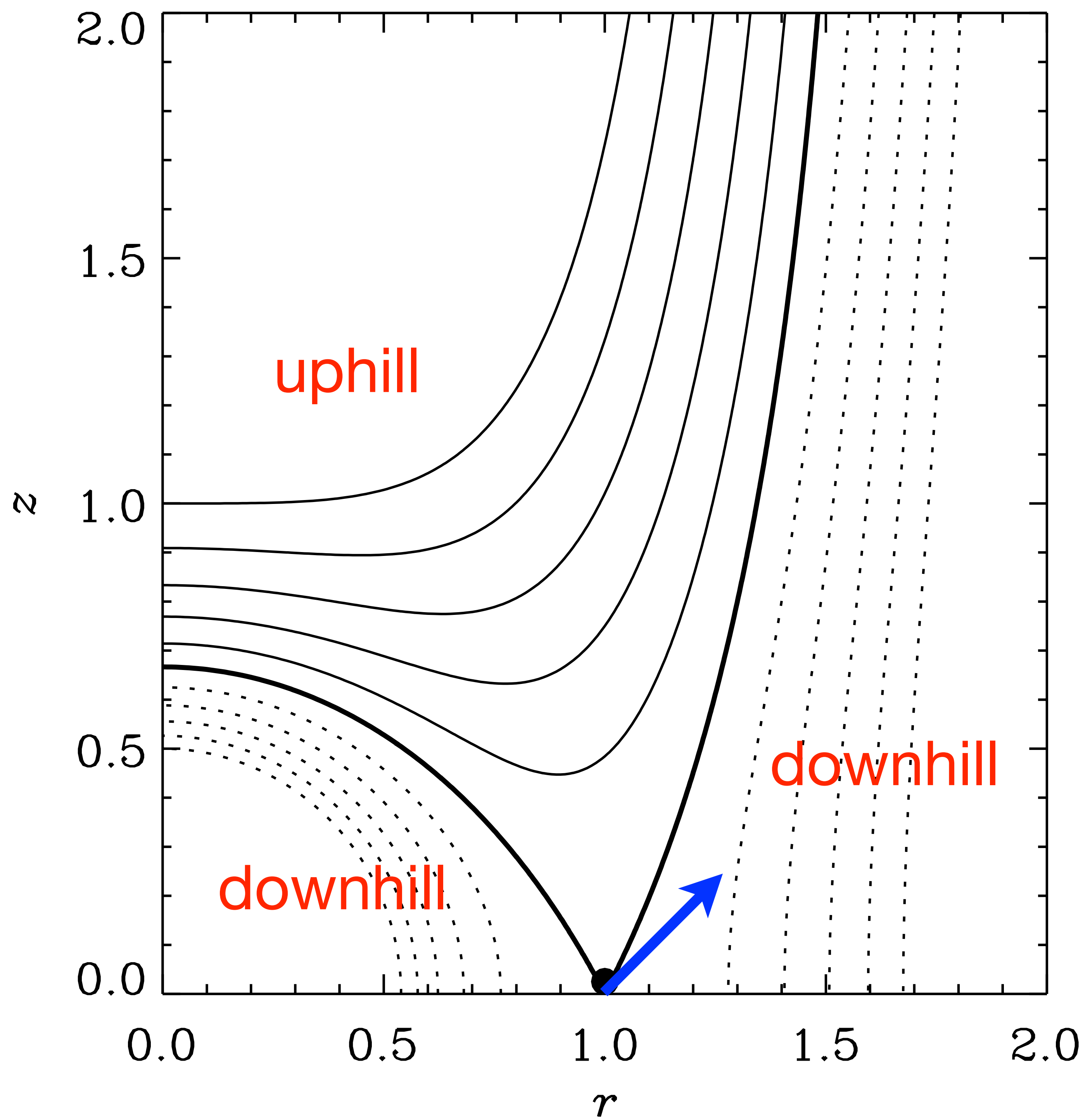
$$z^2 \approx 3(r-1)^2 \qquad \Rightarrow \; z \approx \pm\sqrt{3}(r-1)$$

so saddle point at footpoint

- For $z \gg 1, \quad r \to \sqrt{3}$

Jet launching

$$\Phi^{cg} = -GM(r^2 + z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2$$



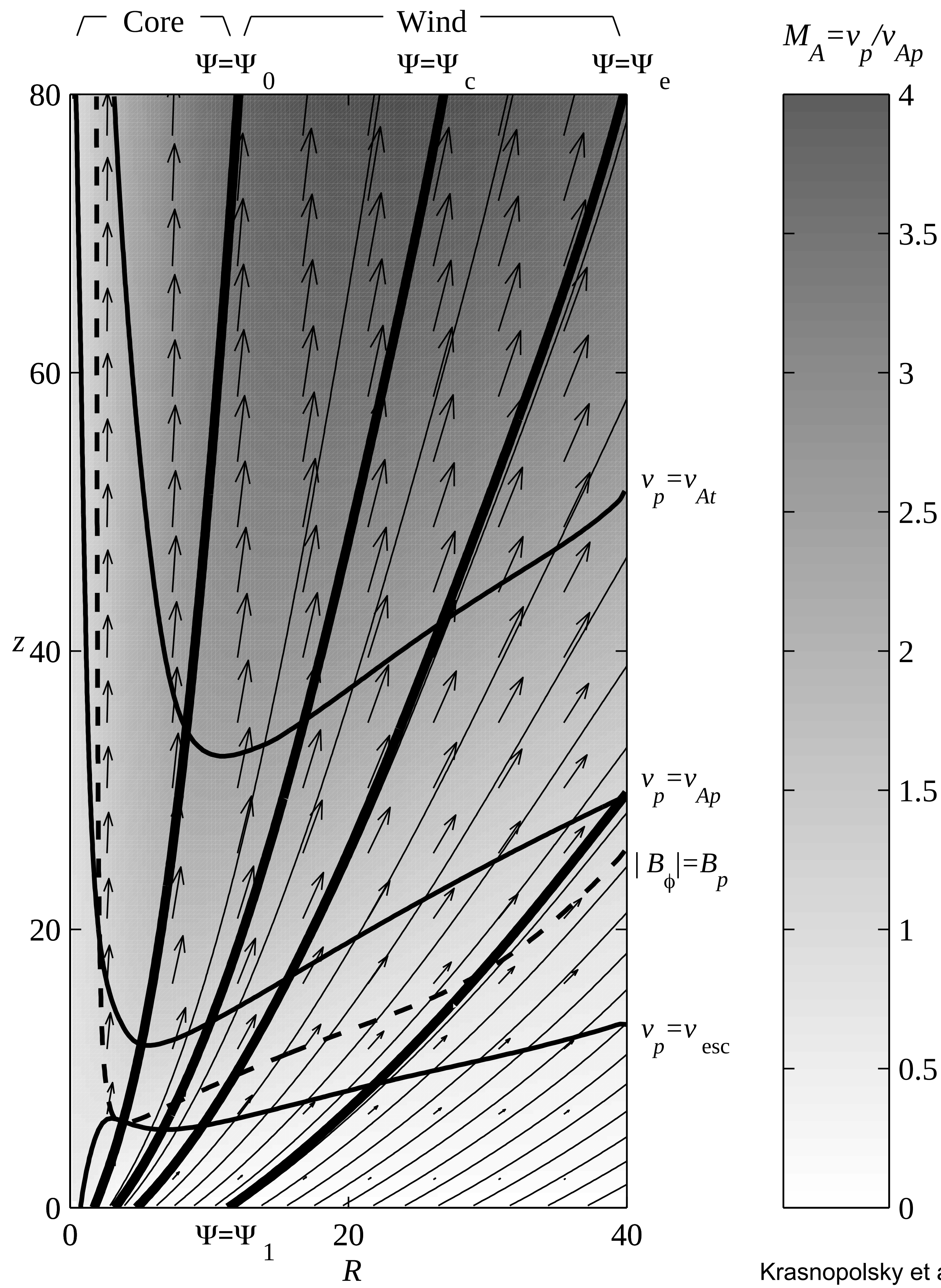
units:
$r_0 = 1$

Jet launching

- If poloidal magnetic field is inclined to the vertical direction by more than 30° and expands sufficiently outwards then the outflow is accelerated mechanically along it
- "Magnetocentrifugal acceleration" (Blandford & Payne 1982)
- Different from stellar winds, which are usually driven thermally
- Thermal assistance is required to accelerate the flow through the slow magnetosonic point close to the surface of the disc (where the mass-loss rate is determined)

Jet launching

- Numerical simulations (Krasnopolsky et al. 1999)



Krasnopolsky et al, 1999: ApJ., 526, 631

Jet launching

- Magnetically driven accretion:
- To allow mass $\Delta M_{\rm acc}$ to be accreted from radius r_0, angular momentum $r_0^2 \Omega_0 \, \Delta M_{\rm acc}$ must be removed
- If mass $\Delta M_{\rm jet}$ is lost in a magnetized outflow from r_0, angular momentum removed is $\ell \, \Delta M_{\rm jet} = r_{\rm a}^2 \Omega_0 \, \Delta M_{\rm jet}$
- So accretion can (in principle) be driven by an outflow, with

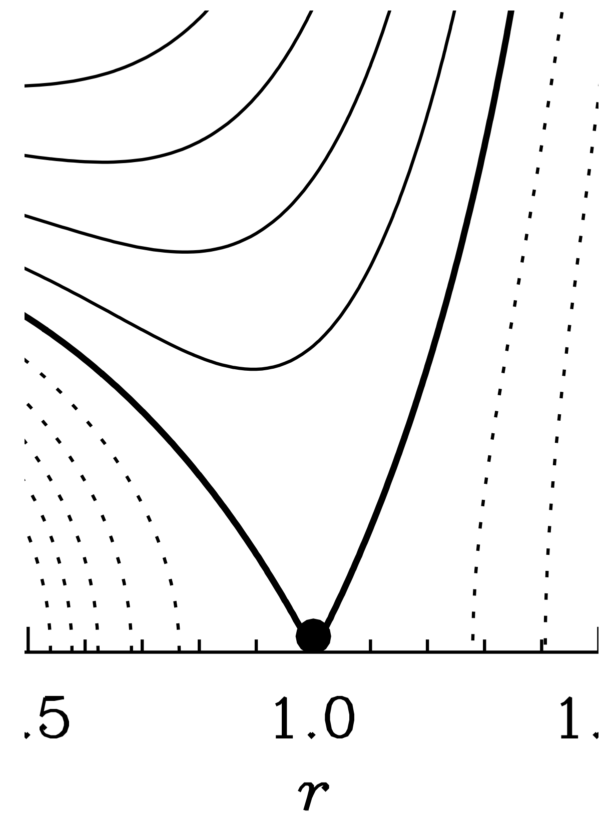
$$\frac{\dot{M}_{\rm acc}}{\dot{M}_{\rm jet}} \sim \frac{r_{\rm a}^2}{r_0^2}$$

- Magnetic lever arm makes this process efficient
- Protostellar systems do show $\dot{M}_{\rm acc}/\dot{M}_{\rm jet} \sim 10$
- Not clear if wind-driven discs can be steady or stable

Jet launching

- Jet launching in the local approximation:
- Effective potential in local approximation:

$$\begin{aligned}\Phi &= -\Omega S x^2 + \tfrac{1}{2}\Omega_z^2 z^2 \\ &= \tfrac{1}{2}\Omega^2(-3x^2 + z^2) \qquad \text{(Keplerian)}\end{aligned}$$



- Fluid forced to rotate with angular velocity Ω (on field line anchored at reference radius) experiences this effective potential
- Reproduces critical inclination angle of 30°

Summary

- Mechanisms of activity and angular momentum transport in astrophysical discs:
  - Viscous transport
  - Hydrodynamic instability
  - Vortex dynamics
  - Gravitational instability
  - Satellite–disc interaction
  - Magnetorotational instability
  - Magnetized outflows

Summary

- Viscous transport
  - Relevant for planetary rings (macroscopic particles)
- Hydrodynamic instability
  - Mostly thought to be absent or ineffective in standard discs (but controversial)
  - Can be present in non-circular or warped discs

Summary

- Vortex dynamics
  - Can be effective if vortices can be produced and maintained
  - Vortices excite density waves that transport angular momentum
  - Production:
    - “Baroclinic instability”
    - “Rossby vortex instability”, etc.
  - Destruction:
    - Elliptical instability, etc.
    - Inward migration
  - May be relevant in protoplanetary discs (also for planet formation)

Summary

- Gravitational instability
  - Occurs in sufficiently massive and cool discs
  - May produce turbulence or fragmentation depending on cooling
  - Relevant for outer parts of protoplanetary discs
    and discs around black holes in active galactic nuclei
  - Also relevant for planetary rings
- Satellite–disc interaction
  - Embedded or external satellites excite waves and induce
    induce angular momentum transport
  - Applications are quite specific and localized

Summary

- Magnetorotational instability
  - Occurs in sufficiently ionized discs
  - Relevant for high-energy (plasma) accretion discs and for sufficiently ionized layers of protoplanetary discs
  - Questions remain over efficiency of dynamo and transport
- Magnetized outflows
  - Probably requires sufficiently ordered and strong magnetic field
  - Applications may be restricted

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