主成分凝結を考慮した準圧縮系での質量収支

本文書では, 主成分凝結を考慮した準圧縮系での質量収支の式

$$\frac{\partial}{\partial t}\int(\rho+\rho_s)dV = \int\frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial\theta'}{\partial x}+w\frac{\partial\theta'}{\partial z}\right)dV - \int\frac{\overline{\rho}}{\overline{\theta}}D_\theta dV \tag{1}$$

を導出する.

状態方程式

$$\rho = \frac{p_0}{R}\frac{\Pi^{c_v/R}}{\theta} \tag{2}$$

の両辺を t で偏微分すると,

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = -\frac{1}{\theta}\frac{\partial\theta}{\partial t}+\frac{c_v}{R}\frac{1}{\Pi}\frac{\partial\Pi}{\partial t} \tag{3}$$

(3) を線形化すると,

$$\frac{1}{\overline{\rho}}\frac{\partial\rho'}{\partial t} = -\frac{1}{\overline{\theta}}\frac{\partial\theta'}{\partial t}+\frac{c_v}{R}\frac{1}{\overline{\Pi}}\frac{\partial\Pi'}{\partial t}. \tag{4}$$

温位の式

$$\frac{\partial\theta'}{\partial t} = -u\frac{\partial\theta'}{\partial x}-w\frac{\partial\theta'}{\partial z}-w\frac{\partial\overline{\theta}}{\partial z}+\frac{1}{\overline{\Pi}}(Q_{cond}+Q_{rad}+Q_{dis})+D_\theta \tag{5}$$

および圧力方程式

$$\begin{aligned}\frac{\partial\Pi'}{\partial t} &= -\frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}^2}\left[\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x}+\frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z}\right] \\ &\quad -\frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}}M_{cond}+\frac{\overline{c}^2}{c_p\overline{\theta}^2\overline{\Pi}}(Q_{cond}+Q_{rad}+Q_{dis})\end{aligned} \tag{6}$$

を (4) に代入すると,

$$\begin{aligned}\frac{\partial\rho'}{\partial t} &= -\frac{\overline{\rho}}{\overline{\theta}}\frac{\partial\theta'}{\partial t}+\frac{c_v}{R}\frac{\overline{\rho}}{\overline{\Pi}}\frac{\partial\Pi'}{\partial t} \\ &= -\frac{\overline{\rho}}{\overline{\theta}}\left[-u\frac{\partial\theta'}{\partial x}-w\frac{\partial\theta'}{\partial z}-w\frac{\partial\overline{\theta}}{\partial z}+\frac{1}{\overline{\Pi}}(Q_{cond}+Q_{rad}+Q_{dis})+D_\theta\right] \\ &\quad +\frac{c_v}{R}\frac{\overline{\rho}}{\overline{\Pi}}\left\{-\frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}^2}\left[\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x}+\frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z}\right]-\frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}}M_{cond}\right. \\ &\quad \left.+\frac{\overline{c}^2}{c_p\overline{\theta}^2\overline{\Pi}}(Q_{cond}+Q_{rad}+Q_{dis})\right\}\end{aligned} \tag{7}$$

となる.

$$\overline{c^2} = \frac{c_p}{c_v}R\overline{\theta}\overline{\Pi} \tag{8}$$

であることを用いて (8) を書き換えると,

$$
\begin{aligned}
\frac{\partial \rho'}{\partial t} &= -\frac{\overline{\rho}}{\overline{\theta}}\left[-u\frac{\partial \theta'}{\partial x} - w\frac{\partial \theta'}{\partial z} - w\frac{\partial \overline{\theta}}{\partial z} + \frac{1}{\overline{\Pi}}(Q_{cond} + Q_{rad} + Q_{dis}) + D_\theta\right] \\
&\quad + \frac{c_v}{R}\frac{\overline{\rho}}{\overline{\Pi}}\left\{ -\frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}^2}\left[\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x} + \frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z}\right] - \frac{\overline{c}^2}{c_p\overline{\rho}\overline{\theta}}M_{cond} \right. \\
&\quad \left. + \frac{\overline{c}^2}{c_p\overline{\theta}^2\overline{\Pi}}(Q_{cond} + Q_{rad} + Q_{dis}) \right\} \\
&= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) + \frac{\overline{\rho}}{\overline{\theta}}w\frac{\partial \overline{\theta}}{\partial z} - \frac{\overline{\rho}}{\overline{\theta}\overline{\Pi}}(Q_{cond} + Q_{rad} + Q_{dis}) - \frac{\overline{\rho}}{\overline{\theta}}D_\theta \\
&\quad - \frac{1}{\overline{\theta}}\left[\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x} + \frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z}\right] - M_{cond} + \frac{\overline{\rho}}{\overline{\theta}\overline{\Pi}}(Q_{cond} + Q_{rad} + Q_{dis}) \\
&= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) + \frac{\overline{\rho}}{\overline{\theta}}w\frac{\partial \overline{\theta}}{\partial z} - \frac{\overline{\rho}}{\overline{\theta}}D_\theta - \frac{1}{\overline{\theta}}\left[\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x} + \frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z}\right] \\
&\quad - M_{cond} \\
&= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) - \frac{\overline{\rho}}{\overline{\theta}}D_\theta \\
&\quad + \frac{1}{\overline{\theta}^2}\frac{\partial \overline{\theta}}{\partial x}\overline{\rho}\overline{\theta}u - \frac{1}{\overline{\theta}}\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x} + \frac{1}{\overline{\theta}^2}\frac{\partial \overline{\theta}}{\partial z}\overline{\rho}\overline{\theta}w - \frac{1}{\overline{\theta}}\frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z} - M_{cond} \\
&= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) - \frac{\overline{\rho}}{\overline{\theta}}D_\theta \\
&\quad - \frac{\partial}{\partial x}\left(\frac{1}{\overline{\theta}}\right)\overline{\rho}\overline{\theta}u - \frac{1}{\overline{\theta}}\frac{\partial(\overline{\rho}\overline{\theta}u)}{\partial x} + \frac{\partial}{\partial z}\left(\frac{1}{\overline{\theta}}\right)\overline{\rho}\overline{\theta}w - \frac{1}{\overline{\theta}}\frac{\partial(\overline{\rho}\overline{\theta}w)}{\partial z} - M_{cond} \\
&= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) - \frac{\overline{\rho}}{\overline{\theta}}D_\theta - \frac{\partial(\overline{\rho}u)}{\partial x} - \frac{\partial(\overline{\rho}u)}{\partial z} - M_{cond}
\end{aligned}
\tag{9}
$$

となる. また基本場の物理量は時間に依存しないので,

$$
\frac{\partial \overline{\rho}}{\partial t} = 0 \tag{10}
$$

である. 一方, 雲密度 ρ_s の時間発展方程式 (雲粒落下無し) は

$$
\frac{\partial \rho_s}{\partial t} = -\frac{\partial(\rho_s u)}{\partial x} - \frac{\partial(\rho_s u)}{\partial z} + M_{cond} \tag{11}
$$

となる. (9), (10), (11) の辺々を加えると, 全密度の時間発展方程式

$$
\begin{aligned}
\frac{\partial(\overline{\rho} + \rho' + \rho_s)}{\partial t} &= \frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z}\right) - \frac{\overline{\rho}}{\overline{\theta}}D_\theta \\
&\quad - \frac{\partial[(\overline{\rho} + \rho_s)]u}{\partial x} - \frac{\partial[(\overline{\rho} + \rho_s)]w}{\partial z}
\end{aligned}
\tag{12}
$$

が得られる. (12) を領域全体にわたって積分すると,

$$\frac{\partial}{\partial t}\int(\overline{\rho}+\rho'+\rho_s)dV = \int\frac{\overline{\rho}}{\overline{\theta}}\left(u\frac{\partial\theta'}{\partial x}+w\frac{\partial\theta'}{\partial z}\right)dV-\int\frac{\overline{\rho}}{\overline{\theta}}D_\theta dV \tag{13}$$

となり, 質量収支の式 (1) が得られる.